# Cyclic Higgs bundles and Labourie's conjecture in rank 2

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January 6th, 2015

**Abstract**

Notes for the expository talk on the proof in [Lab14] of Labourie's conjecture in rank 2, given at the Workshop "Higgs Bundles and Harmonic Maps" in Asheville, NC.

In this talk I will present some recent results by François Labourie in [Lab14] proving a special case of a conjecture of his owns (also stated by Bill Goldman, see [Lab06, Gol10]). These notes are organized as follows: The first section includes some preliminaries on Lie theory and the Hitchin component, mostly to fix notations. In the second section we will introduce Labourie's version of Hitchin's map and state the Conjecture (about it being a homeomorphism); the third section introduces cyclic objects. In the final section we reduce Labourie's conjecture in rank 2 (or, rather, the fact that the map is an immersion, the only part of the statement that is still conjectural) to some more technical but easier statement about cyclic maps, the proof of which we sketch.

**Acknowledgments:** A big thank to the organizers of the workshop (Brian Collier, Qiongling Li and Andy Sanders) for the huge preparation work, the amazing location and the stimulating environment!

# 1 Lie group preliminaries and Hitchin's component

## 1.1 Lie groups fundamentals

The proof of most of these statement will not be part of my talk. To fix notations and signs, I will try to present the following results only by depicting the picture for $\mathrm{SL}_3(\mathbb{R})$, by writing down matrices. Here we focus on (a split real form of) $G^{\mathbb{C}} = \mathrm{SL}_3(\mathbb{C})$ for brevity. Remark that we will not be working literally with $\mathrm{SL}_3(\mathbb{R})$: There are choices that have an effect on every other choice, the usual Cartan algebra and involution give another (isomorphic, of course) split real form $G_0$. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_3(\mathbb{C})$, that has rank 2. We make the usual choice of the (trace-zero) diagonal matrices. For each $\alpha$ in the dual

$\mathfrak{h}^*$ we define the root spaces:

$$\mathfrak{g}_\alpha = \left\{u \in \mathfrak{g}^{\mathbb{C}} \ : \ \forall h \in \mathfrak{h},\ [h,u] = \alpha(h)u\right\}$$

This is either 0 or has dimension one; in the last case we call $\alpha$ a root, and denote by $\Delta$ the set of roots. In the case of $\mathfrak{sl}_3(\mathbb{C})$, considering the basis of $\mathfrak{h}$ given by $h_1 = \mathrm{diag}(1,-1,0)$ and $h_2 = \mathrm{diag}(0,1,-1)$, we have an associated dual basis $\lambda_1$, $\lambda_2$. The set of roots is

$$\Delta = \left\{\alpha_1 = 2\lambda_1 - \lambda_2, \alpha_2 = 2\lambda_2 - \lambda_1, \eta = \alpha_1 + \alpha_2 = \lambda_1 + \lambda_2, -\alpha_1, -\alpha_2, -\eta\right\}.$$

Each of these has a 1-dimensional root space, spanned, in the same order, by $x_{12}, x_{23}, x_{13}, x_{21}, x_{32}, x_{31}$, where $x_{ij}$ is a matrix having zeros everywhere except for a 1 in position $(i,j)$. Remark that $\mathfrak{g}^{\mathbb{C}}$ is the direct sum of $\mathfrak{h}$ and the span of these elements, as it should.

The set of roots is uniquely determined; however, one chooses in a non-unique way a set of positive roots $\Delta^+$, with the property that for every $\alpha \in \Delta$ exactly one between $\alpha$ and $-\alpha$ is in $\Delta^+$ and that the sum of positive roots stays positive. The standard choice is $\alpha_1$, $\alpha_2$ and $\eta$. Given this choice there is a unique subset of simple roots $\Pi$, of cardinality exactly the rank (2 in this case), and such that no simple root is the sum of two other positive roots (here, clearly $\alpha_1$ and $\alpha_2$ are the simple ones in $\Delta^+$). For every root $\alpha$ define the *coroot* $h_\alpha$ as the unique element in the complex line $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(h_\alpha) = 2$ (the alternative definition used by Labourie is equivalent to this). In our case we have $h_{\alpha_1} = h_1$ and $h_{\alpha_2} = h_2$, so we may safely drop the $\alpha$ in the index (although in general, of course, $h_{\alpha_i}$ is well-defined, while our choice of $h_i$ was arbitrary). Furthermore, $h_\eta = h_1 + h_2 = \mathrm{diag}(1,0,-1)$ and $h_{-\alpha} = -h_\alpha$, clearly.

Define a *Chevalley system* following [Bou05], Chapter VIII, §2, Definition 3 (the definition in Labourie's preprint seems unfortunately to have some inconsistent signs) by the following conditions:

$$X_\alpha \in \mathfrak{g}_\alpha, \quad [X_\alpha, X_{-\alpha}] = h_\alpha, \quad \text{the map } \left\{ \begin{array}{rcl} \mathfrak{g} & \to & \mathfrak{g} \\ \mathfrak{h} \ni h & \mapsto & -h \\ X_\alpha & \mapsto & X_{-\alpha} \end{array} \right. \quad \text{is an automorphism.}$$

In our case, $X_{\alpha_1} = x_{12}$, $X_{\alpha_2} = x_{23}$ and $X_\eta = x_{13}$. Remark that trivially $[X_\alpha, X_\beta] \in \mathfrak{g}_{\alpha+\beta}$. So, if $\alpha + \beta \in \Delta$, we can define $N_{\alpha\beta}$ so that $[X_\alpha, X_\beta] = N_{\alpha\beta} X_{\alpha+\beta}$, and these turn out to be integers, more exactly $N_{\alpha\beta}$ is (up to a sign) one plus the biggest integer $q$ such that $\alpha - q\beta$ is a root.

In general, for fixed $\Delta^+$, we denote by $\eta$ the longest root, that is the *unique* positive root such that there is no other positive root $\alpha$ such that $\eta + \alpha$ is again a root. We introduce the following two elementary but fundamental subsets of $\Delta$:

**Definition 1.1.** Fix a Cartan subalgebra and a set of simple roots $\Pi$. The set of *cyclic roots* $Z$ is the union of $-\Pi$ and $\{\eta\}$. The set of *conjugate cyclic roots* $Z^\dagger$ is $-Z$, i.e. the union of symple roots and the negative of the longest root.

## 1.2 Kostant's principal subalgebras

The following is due to Kostant. For every complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, once a set of positive root $\Delta^+$ is fixed, one can define the *principal* $\mathfrak{sl}_2(\mathbb{C})$ in it by:

$$a = \frac{1}{2}\sum_{\alpha\in\Delta^+} h_\alpha =: \sum_{\alpha_i\in\Pi} r_{\alpha_i} h_{\alpha_i}; \quad X = \sum_{\alpha_i\in\Pi} \sqrt{r_{\alpha_i}}x_{\alpha_i}; \quad Y = \sum_{\alpha_i\in\Pi} \sqrt{r_{\alpha_i}}x_{-\alpha_i}$$

In our case, $a = h_\eta = \mathrm{diag}(1, 0, -1)$ (beware that this is just a coincidence!), so that $r_{\alpha_i} = 1$, and $X = x_{12} + x_{23}$, $Y = x_{21} + x_{32}$ are the $3 \times 3$ Jordan blocks with 0 on the diagonal. This generates a subalgebra, isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, in particular $[a, X] = X$, $[a, Y] = -Y$ and $[X, Y] = a$. The following fact is due to Kostant and crucial:

$$\deg(\alpha) = \alpha(a) \text{ for every root } \alpha.$$

Working in a more invariant way, one can define a principal subalgebra to be a copy of $\mathfrak{sl}_2(\mathbb{C})$ containing an element conjugate to $a$, and an $\mathfrak{h}$-principal subalgebra as one such that $a \in \mathfrak{h}$. We will not really have to care about these invariant things, but everything works just fine. Even better, Kostant proved that any two principal subalgebras are conjugated, and any two $\mathfrak{h}$-principal are conjugated by elements of (the Lie subgroup associated to) $\mathfrak{h}$. For us, the most relevant part of Kostant's work is the following decomposition:

**Proposition 1.2** (Kostant)**.** *With notation as above, let $\mathfrak{s}$ be a principal subalgebra. There are well defined (increasing) integers $m_1, \ldots, m_\ell$, where $\ell = \mathrm{rk}(G^{\mathbb{C}})$, called the exponents of $G^{\mathbb{C}}$, such that the Lie algebra decomposes as a direct sum of irreducible representations of $\mathfrak{s}$:*

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{i=1}^{\ell} \mathfrak{v}_i, \quad \dim_{\mathbb{C}}(\mathfrak{v}_i) = 2m_i + 1, \quad \mathfrak{v}_1 = \mathfrak{s}.$$

In particular, the dimensions in rank 2 are easy to compute (since $m_1 = 1$ always and in this case $\dim(G^{\mathbb{C}}) = 2m_2 + 2m_1 + 2$), and actually for $\mathrm{SL}(n, \mathbb{C})$ the exponents are just $m_i = i$. We will also use the other decomposition $\mathfrak{g}^{\mathbb{C}} = \bigoplus_{m=-m_\ell}^{m_\ell} \mathfrak{g}_m$, where $\mathfrak{g}_m$ is the set of elements $u \in \mathfrak{g}^{\mathbb{C}}$ such that $[a, u] = mu$ ($a$ is semisimple, so this always gives a decomposition of the whole of $\mathfrak{g}^{\mathbb{C}}$). Since every $\mathfrak{v}_i$ is a representation of $\mathfrak{s}$, in particular there is an element $e_i \in \mathfrak{v}_i$ of highest weight (defined up to a multiple). By additivity of weights, necessarily $[X, e_i] = 0$. Also, clearly, $e_i \in \mathfrak{g}_{m_i}$. Then every $\mathfrak{v}_i$ is simply spanned by $e_i, \mathrm{ad}_Y(e_i), \ldots, \mathrm{ad}_Y^{2m_i}(e_i)$.

Let's be more explicit in our situation $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_3(\mathbb{C})$, where things are well depicted (actually, every $\mathfrak{sl}_n(\mathbb{C})$ works in the same way). Here every $e_i$ lives on the $i$-th diagonal, and taking $\mathrm{ad}_Y$ "lowers" the diagonal by one. For $n = 2$ there is but one choice (up to multiples):

$$e_1 = X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};\ e_2 = x_{13};\ \mathrm{ad}_Y(e_2) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};\ \mathrm{ad}_Y^2(e_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathfrak{h},$$

et cetera. It is clear that $X_\eta \in \mathfrak{g}_{m_\ell}$ is equal to $e_{m_\ell}$, in all generality.

Now we fix a Cartan involution that keeps $\mathfrak{h}$ invariant. Here there is some ambiguity in general, we choose the standard one: $\rho(u) = -u^*$ (here $*$ denotes the transpose conjugate). This is a compact real form. The Killing form is (up to a multiple) given by $\langle u|v\rangle = \operatorname{trace}(u \cdot v)$. With these choices, one gets the Frobenius Hermitian product by combining the two of them, i.e. $\langle u, v\rangle = \operatorname{trace}(uv^*)$. These objects behave very well with respect to the root decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$, that is orthogonal with respect to $\langle\cdot,\cdot\rangle$. In general, we pick the unique Cartan involution associated to the Chevalley system, i.e. we *define* $X_\alpha^* = X_{-\alpha}$ (it preserves $\mathfrak{h}$, but of course it is not the identity there, since $\mathfrak{h}$ is complex and $*$ is anti-linear). There are two more real forms we need to introduce, linked to each other.

**Proposition 1.3** (Kostant, Labourie). *For fixed $\mathfrak{h}$, $\Delta^+$, an $\mathfrak{h}$-principal $\mathfrak{s}$ and $\rho$ as above, there exists a* unique *$\mathbb{C}$-linear involution $\sigma$ that preserves globally both $\mathfrak{h}$ and $\mathfrak{s}$, such that $\lambda = \sigma \circ \rho = \rho \circ \sigma$, such that the real form $G_0 := \operatorname{Fix}(\lambda)$ is split, $\sigma(h_\eta) = h_\eta$ and $\sigma$ globally preserves $\mathfrak{g}_Z = \bigoplus_{\alpha \in Z} \mathfrak{g}_\alpha$ and $\mathfrak{g}_{Z^\dagger}$.*

Let us see who are these elements in the case of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_3(\mathbb{C})$. As anticipated, $G_0$ will not be the usual $\mathrm{SL}_3(\mathbb{R})$ (hence, $\sigma$ is not the complex conjugation). The construction by Kostant asks: $\sigma(e_i) = -e_i$ for every $i$ as well as $\sigma(Y) = -Y$. In particular, on $\operatorname{ad}^i_Y(e_i)$, $\sigma$ acts as $(-1)^{i+1}$ (for example, $\sigma(a) = \sigma([X, Y]) = a$). These objects together generate $\mathfrak{g}^{\mathbb{C}}$, so that we may decompose $\mathfrak{g}^{\mathbb{C}}$ in eigenspaces for $\sigma$ as follows:

$$\operatorname{Ker}(\sigma + \mathrm{Id}) = \left\{ \begin{pmatrix} a & b & d \\ c & -2a & b \\ f & c & a \end{pmatrix} \right\}, \quad \operatorname{Ker}(\sigma - \mathrm{Id}) = \left\{ \begin{pmatrix} g & h & 0 \\ j & 0 & -h \\ 0 & -j & -g \end{pmatrix} \right\},$$

where $a, b, c, d, f, g, h, j \in \mathbb{C}$. Remark that in particular the fixed points of this need not be half-dimensional. The composition $\lambda = \sigma \circ \rho$, then, has the following set of fixed points:

$$\mathfrak{g}_0 = \operatorname{Ker}(\lambda - \mathrm{Id}) = \left\{ \begin{pmatrix} x + iy & t + v + i(u + w) & r + is \\ t - v + i(w - u) & -2x & t - v + i(u - w) \\ r - is & t + v - i(u + w) & x - iy \end{pmatrix} \right\}, \tag{1}$$

where now $r, s, t, u, v, w, x, y \in \mathbb{R}$. In general, the fixed points of $\sigma$ are the complexification of the maximal compact subalgebra $\mathfrak{k}_0$ of $\mathfrak{g}_0$ (since they are the complexification of $\operatorname{Fix}(\rho) \cap \operatorname{Fix}(\Lambda)$), which in this case must be isomorphic to $\mathfrak{so}(3)$, and thus has dimension 3 as expected.

## 1.3 Intermezzo: Homogeneous spaces and Maurer-Cartan forms

In the following, we will be dealing with two different homogeneous spaces. Although we do not need much of their geometry, I collect here some results we will need to uniformize the notations.

A homogeneous space $X$ is a space on which a Lie group $G$ acts in a smooth and transitive way. Fix a point $x_0 \in X$ and let $C$ be its stabilizer (the set of elements $g \in G$ such that $g \cdot x_0 = x_0$). Then the left multiplication gives an equivariant isomorphism:

$$X \cong G/C.$$

This is the aspect that our homogeneous spaces will always have. Also, we assume $C$ to be a compact subgroup. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{c}$ that of $C$. We will also assume that the homogeneous space is *reductive*, that is, there is an $\mathrm{Ad}(C)$-invariant subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{m}$. We get in particular the following infinitesimal relations:

$$[\mathfrak{c}, \mathfrak{c}] \subset \mathfrak{c}; \quad [\mathfrak{c}, \mathfrak{m}] \subset \mathfrak{m}. \tag{2}$$

Remark that $\mathfrak{m}$ is *not* a Lie-subalgebra, that is, $[\mathfrak{m}, \mathfrak{m}]$ will not be in $\mathfrak{m}$. There is a canonical map

$$\begin{aligned} \mathfrak{g} \times X &\to TX \\ (\xi, x) &\mapsto \frac{\partial}{\partial t}\Big( \exp(t\xi) \cdot x \Big)\Big|_{t=0}. \end{aligned} \tag{3}$$

Since the action is smooth and transitive, this map is surjective. Actually, more is true: We can consider the following decomposition in subbundles of the trivial bundle $X \times \mathfrak{g}$:

$$X \times \mathfrak{g} = [\mathfrak{c}] \oplus [\mathfrak{m}], \text{ where } [\mathfrak{c}]_x = \mathrm{Ad}_x(\mathfrak{c}), \ [\mathfrak{m}]_x = \mathrm{Ad}_x(\mathfrak{m}).$$

Although $x \in X$, these spaces are well defined, since $\mathfrak{c}$ and $\mathfrak{m}$ are $\mathrm{Ad}(C)$-invariant. Then the map (3) has kernel $[\mathfrak{c}]$ and induces a bundle isomorphism between $[\mathfrak{m}]$ and $TX$; this gives a right inverse of this map $TX \to [\mathfrak{m}]$, known as the *Maurer-Cartan form* $\beta \in \Omega^1([\mathfrak{m}]) \subset \Omega^1(\mathfrak{g})$. An extrinsic way to define it is to take a local section $s\colon G/C \to G$ of the projection, and to define:

$$\beta(v) = \pi_{[\mathfrak{m}]}\Big( \mathrm{d}s_x(v) \cdot s(x)^{-1} \Big).$$

The projection onto $[\mathfrak{m}]$ gets rid of the dependence on the chosen section $s$.

There is a trivial flat connection $\mathrm{d}$ on the trivial bundle $X \times \mathfrak{g}$. In general, this does *not* preserve the splitting $[\mathfrak{c}] \oplus [\mathfrak{m}]$. We introduce the canonical connection $\nabla$, defined by:

$$\nabla v = \mathrm{d}v - \mathrm{ad}(\beta)(v), \quad v \in \Omega^0([\mathfrak{m}]).$$

Actually, one can check that the same formula gives the exterior differential $\mathrm{d}^\nabla$ when $v \in \Omega^p([\mathfrak{m}])$ is a $p$-form, where one defines

$$\mathrm{ad}(\beta)(v) = [\beta \wedge v]$$

as the wedge product on the form part and the Lie bracket on the Lie algebra part. There is no harm in extending this to sections of the whole trivial bundle

$X \times \mathfrak{g}$, i.e. to functions $X \to \mathfrak{g}$. The following formulas are straightforward, although not immediate, to check (see for example [BR90], Chapter 1):

$$\begin{aligned} \mathrm{d}\beta &= \big(1 - \frac{1}{2}\pi_{[\mathfrak{m}]}\big)[\beta \wedge \beta] \quad \text{i.e.} \quad \mathrm{d}^{\nabla}\beta = -\frac{1}{2}\pi_{[\mathfrak{m}]}[\beta \wedge \beta]; \\ T^{\nabla} &= \mathrm{d}^{\nabla}\beta = -\frac{1}{2}\pi_{[\mathfrak{m}]}[\beta \wedge \beta]; \\ R^{\nabla} &= -\frac{1}{2}\Big[\big(1 - \pi_{[\mathfrak{m}]}\big)[\beta \wedge \beta] \wedge \beta\Big] = -\frac{1}{2}\Big[\big(2\mathrm{d}^{\nabla}\beta + [\beta \wedge \beta]\big) \wedge \beta\Big], \end{aligned} \tag{4}$$

Before returning to our main subject, let us mention the best-known (and easiest to deal with) of the homogeneous spaces, the *symmetric* ones. These are characterized in a certain way by the existence of an involution preserving $C$; for us, the following infinitesimal relation, that completes (2), will be enough:

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{c}.$$

In this case, the relations in (4) simplify greatly, becoming $\mathrm{d}^{\nabla}\beta = 0$ and $T^{\nabla} = 0$. Also, $R^{\nabla} = -\frac{1}{2}\big[[\beta \wedge \beta] \wedge \beta\big]$. This is the expression for the curvature of the Levi-Civita connection for a symmetric space. One can see that in this case the stabilizer $C$ is a *maximal* compact subgroup $K$ of $G$.

Points of a symmetric space are in bijection with *Cartan involutions*. In more concrete terms, if $G \subset \mathrm{GL}_r(\mathbb{C})$, these are just positive definite Hermitian metrics on $\mathbb{C}^r$ (more precisely, of the form $gg^*$ with $g \in G$). We can always consider the adjoint action $G \subset \mathrm{GL}(\mathfrak{g})$, and in this way we get a metric on $X \times \mathfrak{g}$ as follows:

$$\langle \xi, \eta \rangle_x = \Big\langle \mathrm{Ad}_x^{-1}(\xi) \Big| \big(\mathrm{Ad}_x^{-1}(\eta)\big)^* \Big\rangle. \tag{5}$$

Here $\langle | \rangle$ denotes the Killing form and $*$ the conjugate transpose (in the case of linear groups $G$, otherwise one needs to fix beforehand a Cartan involution, i.e. a base point giving the isomorphism between the symmetric space and $G/K$, as in the beginning). This is well defined by invariance of the Killing form and because $K$ is unitary.

In the general picture at the beginning of this subsection, since we asked $C$ to be compact, it is contained in a maximal compact subgroup $K$. We get a projection map

$$G/C \to G/K$$

that allows us to read, in particular, all of the structure we introduced for $G/K$ on $G/C$, too. In particular, the Hermitian product (5) gives a notion on $\xi^*$ on $\mathfrak{g}$ (which is, at every point, just the opposite of the Cartan involution given by the point). It is then very easy to check that the Maurer-Cartan forms are related as follows:

$$\beta_{G/K} = \frac{1}{2}\big(\beta_{G/C} + \beta^*_{G/C}\big).$$

In the following, as Labourie does, we will write $\omega = \beta_{G/C}$ when $C = T$ is a compact torus.

## 1.4 Hitchin triples

A triple $(\mathfrak{h}, \rho, \lambda)$ as in Proposition 1.3 will be called a *Hitchin triple*. Such objects exist and are unique, essentially by the same proposition. Let $X$ be the set of these triples. The complex group $G^{\mathbb{C}}$ acts on $X$ by conjugation, and one sees easily that the stabilizer of a given point $(\mathfrak{h}, \rho, \lambda)$ is a compact torus $T$ having Lie algebra $\mathfrak{t} = \mathfrak{g}_0 \cap \mathfrak{h} \cap \mathfrak{k}$ (i.e. the intersection of the Cartan subalgebra with the maximal compact subalgebra of the split algebra $\mathfrak{g}_0$ given by $\rho$). Clearly, the rank of $\mathfrak{t}$ is bounded by that of $\mathfrak{h}$, in our case 2. It is known that $\mathfrak{sl}_3(\mathbb{R})$ cannot have a 2-dimensional compact subtorus ("$\mathrm{SL}_3(\mathbb{R})$ is not of Hodge type"), so $\mathfrak{t}$ must have dimension 1 in this case. In our case, by taking only skew-adjoint elements in (1), we see that $\mathfrak{t}$ is generated by $ih_\eta$ i.e. with the notation as in (1), only $y$ can be non-zero. It is also known that the other two rank 2 split real group are of Hodge type, so in those cases $\mathfrak{t}_{\mathbb{C}} = \mathfrak{h}$. Anyway, the upshot is:

$$X \cong G^{\mathbb{C}}/T \text{ is a homogeneous space.}$$

Writing $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{h}_0$, where $\mathfrak{h}_0$ is the orthogonal with respect to $\langle \cdot, \cdot \rangle$ (or $\langle \cdot | \cdot \rangle$, equivalently), by the root spaces decomposition we can identify

$$T_{eT}X \cong \mathfrak{m} = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha .$$

The fundamental point will be that a Hitchin representation gives a *parallel* Hitchin triple, and vice versa. This is just a restatement of Hitchin's construction in [Hit92] §6. More of this in Section 3 and Theorem 3.5. To explain parallelness, as in subsection 1.3, introduce the connection

$$\nabla := D - \mathrm{ad}(\omega), \tag{6}$$

which preserves the decomposition $\mathfrak{g}^{\mathbb{C}} = [\mathfrak{t}] \oplus [\mathfrak{m}]$. The following is elementary but fundamental:

**Lemma 1.4.** *Every $G^{\mathbb{C}}$-equivariant section is automatically $\nabla$-parallel.*

In particular, $\rho$ and $\lambda$ are, as well as the subbundles of $X \times \mathfrak{g}^{\mathbb{C}}$ constructed by taking a $T$-invariant subspace of $\mathfrak{g}^{\mathbb{C}}$ (for example, $\mathfrak{h}$) and pushing it around to construct a bundle (in this case, $[\mathfrak{h}]$, also denoted by $\mathcal{H}$ in Labourie's paper). Together with (4), this gives:

**Corollary 1.5.** *$R^\nabla$ identifies with a $[\mathfrak{t}]$-valued 2-form.*

First of all, the "identifies with a 2-form" is the usual way of seeing the curvature as a 2-form, that in (4) emerges by writing it as the 2-form $F = \mathrm{d}^\nabla \omega + [\omega \wedge \omega]$ wedged with $\omega$ (that is just the identification of $TX$ with $[\mathfrak{m}]$). The rest of the corollary follows at once from the fact that $[\mathfrak{g}_0]$, $[\mathfrak{k}]$ and $[\mathfrak{h}]$ are all invariant (by the above discussion) and self-normalizing (by classic Lie algebra arguments): For every section $\eta$ of $TX$, we have

$$R^\nabla \eta = -\mathrm{d}^\nabla(\mathrm{d}^\nabla \eta),$$

so for example if $\eta \in [\mathfrak{h}]$, then $R^\nabla \eta \in \Omega^2([\mathfrak{h}])$, but also $R^\nabla \eta = [F, \eta]$ so by self-normalization $F \in \Omega^2([\mathfrak{h}])$.

Now let $M$ be a Riemannian manifold, $\tilde{M}$ its universal cover and $\pi_1(M)$ its fundamental group. One sees quickly that it is equivalent to give a (conjugacy class of a) representation $\rho\colon \pi_1(M) \to G$ and a $\rho$-equivariant map $f\colon \tilde{M} \to X$ on the one hand and, on the other hand, a family, for every point $m \in M$, of Hitchin triples, together with a connection $\nabla$ making the triple parallel, plus a 1-form $\omega \in \Omega^1(M, \mathfrak{m})$ such that $D = \nabla + \omega$ is flat.

To conclude the picture about Maurer-Cartan forms, we can construct another homogeneous space, the symmetric space $G^{\mathbb{C}}/K$, that has its own Maurer-Cartan form $\beta$. It is straightforward to see that $\beta = \frac{1}{2}(\omega + \omega^*)$, so that $\nabla$ descends to the Levi-Civita connection on the symmetric space as in subsection 1.3.

Following Labourie's notations, we write

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} \oplus \mathfrak{g}_Z \oplus \mathfrak{g}_{Z^\dagger} \oplus \mathfrak{g}_1, \tag{7}$$

where $\mathfrak{g}_Z$ and $\mathfrak{g}_{Z^\dagger}$ are as in Proposition 1.3, and $\mathfrak{g}_1$ are all the remaining root spaces. Beware that this $\mathfrak{g}_1$ has nothing to do with the one appearing in Kostant's decomposition $\mathfrak{g}^{\mathbb{C}} = \bigoplus \mathfrak{g}_m$! The one in that decomposition is actually the root space of simple roots, so $\mathfrak{g}_{m=1} \subset \mathfrak{g}_{Z^\dagger}$. In the case of $\mathfrak{sl}_3(\mathbb{R})$, $\mathfrak{g}_1$ (as in (7)!) is zero, but this is specific to this setting. Accordingly, we will write $\pi_0, \pi_1, \pi = \pi_Z, \pi^\dagger = \pi_{Z^\dagger}$ and $\omega_0$, $\omega_1$, $\phi = \omega_Z$ and $\phi^\dagger = \omega_{Z^\dagger}$.

## 1.5 Hitchin's section

Let $\Sigma$ be a Riemann surface. In these notes, by Higgs bundle we mean a slightly different notion than usual: If $(P, \Phi)$ is a $G$-Higgs bundle, here we consider the *adjoint* Higgs bundle $(\mathrm{Ad}(P), \mathrm{ad}(\Phi))$. In particular, this corresponds to representations in the adjoint group, the bundle is always flat, and is actually a bundle in the Lie algebra $\mathfrak{g}$. Giving such a Higgs bundle $(\mathcal{G}, \Phi)$ together with a solution to the self duality equations $(\nabla, \rho)$ is thus equivalent to giving a representation $\rho\colon \pi_1(\Sigma) \to G$ and $\rho$-equivariant a harmonic map

$$h\colon \tilde{\Sigma} \to G/K.$$

Indeed, in one direction, over $G/K$ we can consider the flat Lie algebra bundle $G/K \times \mathfrak{g}$; if we pull back all the structure (i.e. Maurer-Cartan form, Levi-Civita connection, etc.) to the trivial bundle $\tilde{\Sigma} \times \mathfrak{g}$ and project it down to a $\mathfrak{g}$-bundle on $\Sigma$, we get the data needed in order to construct a flat bundle $\mathcal{G}$ with a harmonic metric, hence a Higgs bundle $(\mathcal{G}, \Phi)$. This is the point of view that will be most useful. Recall that the self duality equations can be written as

$$\mathrm{d}^\nabla \Phi = 0; \quad \mathrm{d}^\nabla \Phi^* = 0; \quad R^\nabla = [\Phi \wedge \Phi^*].$$

Recall that in the case of $G^{\mathbb{C}} = \mathbb{P}\mathrm{SL}_2(\mathbb{C})$, Hitchin constructed a Higgs bundle on the rank 2 vector bundle $V = K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}}$. Given a solution to the self duality equation as above, consider the totally geodesic map given by a principal $\mathfrak{sl}_2(\mathbb{C})$ inside $\mathfrak{g}^{\mathbb{C}}$, that gives a harmonic map

$$h\colon \tilde{\Sigma} \to \mathbb{P}\mathrm{SL}_2(\mathbb{C})/\mathbb{P}\mathrm{SO}(2) \hookrightarrow G^{\mathbb{C}}/K.$$

Kostant's decomposition $\mathfrak{g}^{\mathbb{C}} = \bigoplus \mathfrak{v}_i$ corresponds to the decomposition of the pull-back of the Lie algebra bundle described above:

$$\mathcal{G} = \bigoplus_{i=1}^{\ell} S^{2m_i}(V) = \bigoplus_{i=1}^{\ell} K^{-m_i} \oplus \cdots \oplus K^{m_i}.$$

The other decomposition we mentioned may be written as $\mathcal{G} = \bigoplus_{m=-m_\ell}^{m_\ell} [\mathfrak{g}_m]$. The amazing fact is that we can relate the two very precisely:

$$[\mathfrak{g}_m] = \mathfrak{g}_m \otimes K^m.$$

Otherwise said, a "fixed" element of $\mathfrak{g}_m$ gives a section of $[\mathfrak{g}_m] \otimes K^{-m}$. Recall that by Kostant's rule $\deg(\alpha) = \alpha(a)$; in particular, $\alpha(a) = 1 \iff \alpha$ is simple. And $Y$ is in $\mathfrak{g}_{m=1}$ (in Kostant's decomposition), hence it gives a section of

$$[\mathfrak{g}_{-1}] \otimes K \subset \mathcal{G} \otimes K;$$

this is exactly the Higgs field constructed by Hitchin and that corresponds to a Fuchsian representation. To get the whole Hitchin section, we simply use all the other highest weight vectors $e_1, \ldots, e_\ell$: For every holomorphic differential $q = (q_1, \ldots, q_\ell)$, with $q_i \in H^0(\Sigma, K^{m_i+1})$, define the Higgs bundle by

$$H(q) = \Big(\mathcal{G}, \Phi = Y + \sum_{i=1}^{\ell} e_i \otimes q_i\Big). \tag{8}$$

Hitchin's theorem is then that this gives a stable Higgs bundle, that is actually a $G_0$-Higgs bundle and a whole connected component of the moduli space. Also, the map $H$ from the Hitchin base $B = \bigoplus_{i=1}^{\ell} H^0(\Sigma, K^{m_i+1})$ to this component (which we denote $\mathrm{Hit}(\Sigma, G_0)$) is a homeomorphism.

# 2 Equivariant Hitchin map and Labourie's conjecture

The Hitchin section above has a main flaw: It is highly dependent on the complex structure $J$ on $\Sigma$. This means that it does not allow to study the action on the Hitchin component of the mapping class group of $\Sigma$. The proposed solution to this problem is to consider the following slight variation of Hitchin's map: Instead of fixing $J$ and considering differential forms that are holomorphic with respect to it, consider the space of holomorphic differential as a bundle over the Teichmüller space $\mathcal{T}$. More exactly, we consider the vector bundle $\mathcal{E} \to \mathcal{T}$ whose fiber over $J$ is given by

$$\mathcal{E}_J = H^0\big(\Sigma, K_J^{m_2+1}\big) \oplus \cdots \oplus H^0\big(\Sigma, K_J^{m_\ell+1}\big).$$

Remark that we have dropped the differentials of order $m_1 + 1 = 2$, corresponding to the terms that in the Higgs field (8) should pair with $e_1 = X$; the total space of the resulting vector bundle $\mathcal{E}$ has thus the same dimension as $B$ (since the missing space of quadratic differentials has the same dimension $6g - 6$ as the Teichmüller space). We can now define Labourie's version of the Hitchin section:

$$\begin{aligned} \Psi\colon \mathcal{E} &\to \mathrm{Hit}(\Sigma, G_0) \\ \big(J, q_{m_2}, \dots, q_{m_\ell}\big) &\mapsto H_J\big(0, q_{m_2}, \dots, q_{m_\ell}\big), \end{aligned}$$

where $H_J$ is the same thing as $H$ in (8), we are only making the complex structure $J$ explicit here. Then we have the following:

**Conjecture 2.1** (Labourie [Lab06], see also [Gol10])**.** *The map $\Psi$ is a homeomorphism.*

The surjectivity of this map has been proved by Labourie in [Lab06] without any assumption on the rank. The injectivity is actually already known for $G_0 = \mathbb{P}\mathrm{SL}_2(\mathbb{R})$ (in this case, it follows easily from Hitchin's paper on self-duality equations) and $G_0 = \mathbb{P}\mathrm{SL}_3(\mathbb{R})$; the paper we are currently discussing proves in particular the injectivity for $G_0$ of rank 2, that is, $\mathbb{P}\mathrm{SL}_3(\mathbb{R})$, $\mathbb{P}\mathrm{Sp}(4, \mathbb{R})$ and the exceptional group $G_2$. A way to restate the conjecture is the following:

**Conjecture 2.2.** *Given a Hitchin representation $\rho\colon \pi_1(\Sigma) \to G_0$ there exists a unique $\rho$-equivariant* minimal *harmonic mapping.*

The link between the two is the following: Given a harmonic map $h$, the associated Higgs field $\Phi$ at a point $x \in \Sigma$ is simply the $(1, 0)$-part of

$$\mathrm{d}_{\mathbb{C}} h_x \colon T_x^{\mathbb{C}}\Sigma \to T_{h(x)}^{\mathbb{C}} G/K \overset{\beta}{\cong} [\mathfrak{p}]_{f(x)} \otimes \mathbb{C} \subseteq \mathfrak{g}^{\mathbb{C}},$$

where $[\mathfrak{p}] \subset G/K \times \mathfrak{g}$ is the subbundle of self-adjoint elements (with respect to $\langle \cdot, \cdot \rangle$) and the isomorphism is given by the right Maurer-Cartan form $\beta$ of $G/K$. The Hopf differential of $h$ is then defined as $\langle \Phi | \Phi \rangle = \mathrm{trace}(\Phi^2)$. This is clearly a quadratic differential, and actually the only one, up to multiple constants. By Donaldson, harmonic maps with vanishing Hopf differential are the same thing as minimal maps. So, under the Hitchin correspondence, admitting a minimal harmonic map is the same thing as coming from a holomorphic differential of the form $(0, q_{m_2}, \dots, q_{m_\ell})$, as in Conjecture 2.1.

# 3 Cyclic Higgs bundles and cyclic maps

We present here the main characters of the paper. Cyclic Higgs bundles are those arising from the Hitchin map $\Psi$ as follows:

**Definition 3.1.** Let $(\Sigma, J)$ be a Riemann surface. A Higgs bundle $(\mathcal{G}, \Phi)$ on $\Sigma$ is a *cyclic Higgs bundle* if it is the image under $H_J$ of a holomorphic differential of the form

$$\mathbf{q} = \big(0, 0, \dots, 0, q_{m_\ell}\big).$$

Recall the notations of Section 1: we fix a Cartan subalgebra $\mathfrak{h}$, a principal subalgebra of $\mathfrak{g}^{\mathbb{C}}$ and a split real form $G_0$ of $G^{\mathbb{C}}$. We get in particular the notion of simple and cyclic roots, and the decomposition of the Maurer-Cartan form $\omega = \omega_0 + \omega_1 + \phi + \phi^{\dagger}$. The second main character is the following

**Definition 3.2.** Let $\Sigma$ be a topological surface (not necessarily closed). A *cyclic map* is a map

$$f \colon \Sigma \to X = G^{\mathbb{C}}/T$$

such that $f^*\omega_0 = f^*\omega_1 = f^*(\phi \wedge \phi) = f^*(\phi^* - \phi^{\dagger}) = f^*(\bar{\omega} - \omega) = 0$, and, furthermore, for every simple root $\alpha \in \Pi$, $f^*(\omega_\alpha)$ is nowhere vanishing.

*Remark* 3.3. Recall that we denoted $\eta^* = -\rho(\eta)$ for the adjoint of $\eta$. Also, we write $\bar{\omega} = \lambda(\omega)$ to mean the conjugation with respect to the fixed real form $G_0$.

Before seeing how these two objects are related, we state the following proposition, essentially due to Baraglia, that will be fundamental.

**Proposition 3.4** (Baraglia)**.** *Let $(\mathcal{G}, \Phi)$ be a cyclic Higgs bundle and $(\nabla, \rho)$ a solution to the self duality equation (here, $\nabla$ is a connection on $\mathcal{G}$, and $\rho$ is a Cartan involution at every fiber of $\mathcal{G}$). In particular, by Hitchin's work, this automatically gives rise to a subbundle in Cartan subalgebras $[\mathfrak{h}] \subset \mathcal{G}$, a family of Hitchin triples and all of the above structure. Then $[\mathfrak{h}]$ is $\nabla$-parallel.*

*Proof.* This follows from the uniqueness of the solutions to the self-duality equation (i.e. the stability of the Higgs bundle by Hitchin). Indeed, consider the automorphism $\phi$ of $\mathcal{G}$ that, under the decomposition $\mathcal{G} = \bigoplus_{m=-m_\ell}^{m_\ell} [\mathfrak{g}_m]$, acts on every $[\mathfrak{g}_m]$ by multiplication by $\zeta_{m_\ell+1}^m$, where $\zeta_{m_\ell+1}$ is a primitive $m_\ell+1$-th root of the unity. The peculiarity of cyclic Higgs bundles is that $\phi(\Phi) = \zeta_{m_\ell+1}^{-1}\Phi$, since it has only terms in $[\mathfrak{g}_{-1}]$ and $[\mathfrak{g}_{m_\ell}]$. It follows that $\phi^*\nabla = \nabla$, since both solve the self duality equation. As $[\mathfrak{h}] = \mathrm{Ker}(\phi)$, the conclusion follows. □

**Theorem 3.5.** *On a closed surface, the data of a cyclic map is equivalent to the data of a complex structure and a cyclic Higgs bundle. More precisely, given a cyclic map on a (possibly open) surface $f \colon \Sigma \to X$, there exists a unique compatible complex structure and the composition $h \colon \Sigma \xrightarrow{f} G^{\mathbb{C}}/T \to G^{\mathbb{C}}/K$ is harmonic and gives rise to a cyclic Higgs bundle. Conversely, given a closed Riemann surface and a cyclic Higgs bundle, there exists a unique compatible cyclic map lifting the solution to the self duality equations.*

*Sketch of proof.* Assume you are given a cyclic map. The claim is that there is a unique complex structure making $\Phi = f^*\phi$ (our candidate for the Higgs field) of type $(1,0)$. This follows from $f^*(\phi \wedge \phi) = 0$ and $f^*\omega_\alpha$ being non-zero: Indeed, the former in particular implies $f^*(\omega_\alpha \wedge \omega_\beta) = 0$ for every roots $\alpha, \beta \in -\Pi$. We may assume that $\alpha + \beta \in \Delta$, i.e. $[x_\alpha, x_\beta] \neq 0$. Now write $f^*\omega_\alpha = \hat{\omega}_\alpha x_\alpha$, and similarly for $\beta$. Let $J_\alpha$ be a complex structure making $f^*\omega_\alpha$ of type $(1,0)$ and compute:

$$0 = (f^*\omega_\alpha \wedge f^*\omega_\beta)(Y, J_\alpha Y) = \Big(\hat{\omega}_\alpha(Y)\hat{\omega}_\beta(J_\alpha Y) - i\hat{\omega}_\alpha(Y)\hat{\omega}_\beta(Y)\Big)[x_\alpha, x_\beta].$$

Since $f^*\omega_\alpha$ never vanishes, $J_\alpha$ makes $\omega_\beta$ of type $(1,0)$. This proves uniqueness, but actually also existence (after proving, in the same way, that $f^*\omega_\eta$ is of type $(1,0)$, as well). Then we know that $\Phi^* = \Phi^\dagger$, so this is of type $(0,1)$ automatically. To prove that the projection to $G^{\mathbb{C}}/K$ is harmonic, one simply considers the curvature equation (4) and projects it: The projection to $[\mathfrak{h}]$ gives $R^\nabla = [\Phi \wedge \Phi^*]$, the projection to $[\mathfrak{g}_Z]$ gives $\mathrm{d}^\nabla \Phi = 0$ (since $\mathrm{d}^\nabla$ preserves all the subspaces like $[\mathfrak{g}_Z]$, by Lemma 1.4), and the one to $[\mathfrak{g}_{Z^\dagger}]$ gives $\mathrm{d}^\nabla \Phi^* = 0$. The fact that the Higgs bundle thus obtained is cyclic is just a consequence of $f^*\omega_0 = f^*\omega_1 = 0$.

Conversely, given a cyclic Higgs bundle on a closed Riemann surface, we can consider the associated solution to the self duality equation $(\nabla, \rho)$. This gives rise to a Hitchin triple that, by Proposition 3.4, is parallel. We already noticed that the data of a parallel Hitchin triple and a "Maurer-Cartan form" $\Omega$ orthogonal to $[\mathfrak{h}]$ and making $\nabla + \Omega$ flat is always induced by a map $f\colon \tilde{\Sigma} \to X = G^{\mathbb{C}}/T$, equivariant with respect to the representation corresponding to the Higgs bundle via the solution of the self duality equations. It is straightforward to check that this map is cyclic. The only slightly tricky part is that for every $\alpha \in \pm\Pi$ then $f^*\omega_\alpha$ never vanishes: This is due to the fact that by the construction of Hitchin, plus the fact that $e_1$ does not appear since $q_2 = 0$, the only contribution to those are given by $Y \in \mathfrak{g}_{-1} = [\mathfrak{g}_{-1}] \otimes K$. In particular, this is a fixed element, and it vanishes at some point if and only if it vanishes everywhere. Moreover, it has a non-zero component in every $\mathfrak{g}_\alpha$ for $\alpha \in -\Pi$, as required. □

# 4 Proof of the Labourie conjecture in rank 2

In this section, we sketch Labourie's proof of the conjecture in the case of rank 2 groups; actually, we will not really prove that $\Psi$ is an homeomorphism, but only that it is (surjective and) an immersion. This is the main step of the proof, the conclusion follows by a general theorem in differential calculus, that we shall omit. Remark that in rank 2 every Higgs bundle corresponding to a minimal map (i.e. with vanishing quadratic differential) is automatically cyclic. Because of this, the conjecture follows from:

**Theorem 4.1.** *The restriction of $\Psi$ to the subbundle of $\mathcal{E}$ having fiber over $J$ given by $H^0\big(\Sigma, K_J^{m_\ell+1}\big)$ is an immersion.*

To prove this, we want to consider a family $J_t$ of holomorphic structures on $\Sigma$ and $q_t$ of holomorphic differentials of degree $m_\ell + 1$; then, construct the associated Higgs bundles $(\mathcal{G}_t, \Phi_t)$ as $H_{J_t}(0, \ldots, 0, q_t)$; finally, prove that if the first order of these Higgs bundles vanishes, then also $\dot{J}_0$ and $\dot{q}_0$ vanish (where we denote $\dot{\eta}_0 = \frac{\partial \eta_t}{\partial t}\big|_{t=0}$ whenever this makes sense).

*Remark* 4.2. Thanks to Theorem 3.5, we get a family of cyclic maps $f_t\colon \tilde{\Sigma} \to X$ which are $\rho_t$-equivariant. Since the moduli spaces of representations and Higgs bundles are homeomorphic, and since we are assuming that the first order of the Higgs bundles vanish, also $[\dot{\rho}_0] = 0$ (actually, here we are using the, much easier, isomorphism between the tangent spaces of the moduli spaces). Differentiating in $t$ the equation

$$f_t(\gamma\tilde{x}) = \rho_t(\gamma) f_t(\tilde{x}),$$

(and choosing the adequate representatives of conjugacy classes) we obtain that $\dot{f}_0(\gamma\tilde{x}) = \rho_0(\gamma)\dot{f}_0(\tilde{x})$, that is, $\xi = \omega(\dot{f}_0(\gamma))$ is a section of the adjoint bundle $\mathrm{Ad}(\rho_0)$ (that has fiber $\mathfrak{g}$). This will be crucial in computations, since dealing with sections of vector bundles we can integrate by parts.

The Theorem is proved by showing that $\dot{J}_0 = \dot{q}_0 = 0$ and also $\xi = 0$. We start by some remarks toward the latter identity. According to the decomposition (7), write

$$\xi = \zeta_0 + \zeta_1 + \zeta + \zeta^\dagger.$$

**Lemma 4.3.** *To show that $\xi = 0$, it is enough to prove that $D\xi = 0$, where $D$ is the flat connection on $\mathrm{Ad}(\rho_0)$.*

*Proof.* This follows from the irreducibility of Hitchin representations (see [Lab06]): Since $\xi$ is $\mathrm{Ad}(\rho_0)$-equivariant, if $D\xi = 0$ we would have a global section of the local system $\mathrm{Ad}(\rho_0)$. Equivalently, this is a fixed element $\xi \in \mathfrak{g}$ commuting to the whole of $\rho_0(\pi_1(\Sigma))$. But the non-existence of such objects is exactly the definiton of an irreducible representation. □

We start by the following reduction:

**Proposition 4.4.** *To prove Theorem 4.1, it suffices to prove that $\xi = \omega(\mathrm{d}f_0(\nu))$, for some vector field $\nu$ on $\Sigma$.*

*Proof.* This is implicit in Labourie's preprint. The first thing to notice is that if $\xi = \omega(\mathrm{d}f_0(\nu))$, then it is in particular self-adjoint: This is because, since $f^*\omega_0 = f^*\omega_1 = 0$, $\omega \circ \mathrm{d}f = \Psi_0 = \Phi_0 + \Phi_0^*$, which takes values in the self-adjoint part of $X \times \mathfrak{g}$. The first step is to prove $\dot{J}_0 = 0$. To do that, recall that by the proof of Theorem 3.5, the definition of $J_t$ is:

$$\Psi_t(J_t v) = i\Phi_t(v) - i\Phi_t(v)^* = \big(i\pi_Z - i\pi_{Z^\dagger}\big)\Psi_t(v) \quad \forall v \text{ (real) vector field on } \Sigma.$$

Consider these as objects on $\Sigma \times (-1, 1)$. Then we can pullback the connection $\nabla = D - \mathrm{ad}(\omega)$ introduced in (6) via $f_t$ to get a connection on the pull-back bundle. Take the covariant derivative with respect to $\frac{\partial}{\partial t}$, that has the fundamental property of commuting with the projections to $Z$ and $Z^\dagger$:

$$\frac{D}{\partial t}\Psi_t(J_t X)\Big|_{t=0} = \big(i\pi_Z - i\pi_{Z^\dagger}\big)\frac{D}{\partial t}\Psi_t(X)\Big|_{t=0}. \tag{9}$$

The crucial fact here is that all these objects live in the pull-back of the bundle $[\mathfrak{p}]$. An elementary lemma in differential geometry (see [dC92] Chap. 3, Lemma 3.4 and Chap. 4, Lemma 4.1) allows us to exchange the differentiations with respect to $\frac{\partial}{\partial t}$ and $X$ (since these two vector fields clearly commute), paying the price of adding the torsion of $\nabla$ in the game; using the torsion formula in (4), one has

$$\Psi_0(\dot{J}_0 v) + \mathrm{d}^{\nabla}_{J_0 v}(\xi) - \pi_{[m]}\big[\xi, \Psi_0(J_0 v)\big] = \big(i\pi_Z - i\pi_{Z^\dagger}\big)\Big(\mathrm{d}^{\nabla}_v(\xi) - \pi_{[\mathfrak{m}]}\big[\mu, \Psi_0(v)\big]\Big).$$

Now since $\xi = \frac{\partial f_t}{\partial t}\big|_{t=0}$ is in $\mathrm{d}f(T\Sigma) \subset [\mathfrak{p}]$, we can only consider the part of this equation that takes values in $[\mathfrak{p}]$; as both $\eta \mapsto \pi_{[\mathfrak{m}]}\eta$ and $\eta \mapsto (i\pi_Z - i\pi_{Z^\dagger})$ commute with the Cartan involution, we obtain:

$$\mathrm{d}^{\nabla}_{J_0 X}\xi + \Psi_0(\dot{J}_0 X) = i\mathrm{d}^{\nabla}_X \zeta - i\mathrm{d}^{\nabla}_X(\zeta^\dagger).$$

Rearranging, we get

$$\Psi_0 \circ \dot{J}_0 = 2i\bar{\partial}\zeta - 2i\partial\zeta^\dagger.$$

To prove $\dot{J}_0 = 0$, it suffices to show that this vanishes as $\Psi_0$ is injective (indeed, for every simple root $\alpha$ already $\pi_\alpha \circ \Psi_0$ is an isomorphism by Definition 3.2). But it is indeed the case that $\zeta$ is holomorphic: In local holomorphic coordinates, the self-duality equation $\mathrm{d}^{\nabla}\Phi_0 = 0$ together with the hypothesis $\zeta = \Phi_0(\nu)$ imply this easily:

$$\frac{D}{\partial\bar{z}}\big(\Phi_0(\nu)\big) = \big(\mathrm{d}^{\nabla}\Phi\big)\Big(\frac{\partial}{\partial\bar{z}}, \nu\Big) - \mathrm{d}^{\nabla}_{\nu}\Phi_0\big(\frac{\partial}{\partial\bar{z}}\big) = 0.$$

We now show that $\xi = 0$, and $\dot{q}_0 = 0$ will follow. Since $\xi = \xi^*$, letting $h_t \colon \tilde{\Sigma} \to G/K$ be the compositions of $f_t$ with the projection, we also have $\xi = \beta(\frac{\partial h_t}{\partial t}\big|_{t=0})$, where $\beta$ is the Maurer-Cartan form of $G/K$. These maps are harmonic, so we can differentiate the harmonic relation $\mathrm{d}^{\nabla}\big(\mathrm{d}h_t \circ J_t\big) = 0$ as in [Spi14]. Using the same symmetry relations as above, together with the expression for the curvature of $\nabla$, for every tangent fields $T, Z$ on $\Sigma$ we get:

$$\begin{aligned}\mathrm{d}^{\nabla}_T\mathrm{d}^{\nabla}_{J_0 Z}\xi - \mathrm{d}^{\nabla}_Z\mathrm{d}^{\nabla}_{J_0 T}\xi - \Big[\Psi_0(T), [\Psi_0(J_0 Z), \xi]\Big] + \Big[\Psi_0(Z), [\Psi_0(T), \xi]\Big]& \\ = -\mathrm{d}^{\nabla}_T\Psi_0\Big(\dot{J}_0 Z\Big) + \mathrm{d}^{\nabla}_Z\Psi_0\Big(\dot{J}_0 T\Big).&\end{aligned} \tag{10}$$

Now $\dot{J}_0 = 0$, hence taking $Z = J_0 T$ one obtains that the left hand side (which is known as the "Jacobi operator" $\mathcal{J}(\xi)$) vanishes. By [Spi14], Proposition 2.5, (4), $\mathcal{J}(\xi) = -\mathrm{d}^*\mathrm{d}\xi$, where $\mathrm{d}^*$ is the adjoint of $\xi$ with respect to $\langle\cdot,\cdot\rangle$. Explicitly, this means that for every $\mathrm{Ad}(\rho)$-equivariant $\eta$,

$$\int_\Sigma \langle \mathrm{d}\xi, \mathrm{d}\eta\rangle \,\mathrm{d}z \wedge \mathrm{d}\bar{z} = 0;$$

then taking $\eta = \xi$ gives $\|\mathrm{d}\xi\|^2 = 0$, hence $\xi = 0$ by Lemma 4.3.

It is easy to conclude that $\dot{q}_0 = 0$: Essentially, $q_t = \mathrm{trace}\big(\Phi_t^{m_\ell}\big)$, and differentiating this expression in $t$ gives a linear expression in $\xi$, hence zero. □

*Remark* 4.5. 1. It is tempting to differentiate the equation $\Phi_t = Y + e_{m_\ell} \otimes q_t$ to deduce directly that $\dot{q}_0 = 0$, since $Y$ and $e_{m_\ell}$ are covariant constant. However, $\dot{\Phi}_0 = 0$ only means that $(\mathcal{G}_t, \Phi_t)$ is constant (to the first order) up to a gauge transformation, so more care would be needed to make this idea work.

2. The proof of the Proposition makes it clear that it is enough to prove that $\xi$ is self adjoint and that $\dot{J}_0 = 0$. Actually, already knowing that $\xi = \xi^*$ would give several advantages, and this can be made precise by proving the following:

**Corollary 4.6.** *We have* $\zeta_0 = 0$.

*Proof.* The main point is that $\zeta_0$ is self adjoint. Indeed, $\zeta_0 \in [\mathfrak{h}]$ by definition, and $\bar{\zeta}_0 = \zeta_0$ (i.e. $\zeta_0 \in [\mathfrak{g}_0]$) since $\xi \in [\mathfrak{g}_0]$, and since the conjugation $\lambda$ preserves the decomposition (7). Hence, the projection to $[\mathfrak{k}]$ of $\zeta_0$ would be in $[\mathfrak{t}]$, but $\zeta_0 = \omega_0(\xi)$ lives in the orthogonal of $[\mathfrak{t}]$.

By the above reasoning, we get directly that $\mathcal{J}(\zeta_0)$ must equal the projection to $[\mathfrak{g}_0]$ on the right hand side of (10). This, however, is obtained as covariant derivatives of $\Psi_0$, that takes values in $[\mathfrak{g}_Z \oplus \mathfrak{g}_{Z^\dagger}]$. We thus conclude by remarking that $\mathcal{J}$ preserves the decomposition (7). This is not completely trivial, but it can be deduced through straightforward computations using the different expressions for $\mathcal{J}$: Write, in local orthonormal coordinates, $\Phi(z) = \nu(z)\mathrm{d}z$. Then, writing $\mathrm{d}^\nabla = \partial^\nabla + \bar{\partial}^\nabla$, we get:

$$\begin{aligned}\mathcal{J}(\eta) - 2\partial^\nabla\bar{\partial}^\nabla(\eta) - 2\bar{\partial}^\nabla\partial^\nabla(\eta) &= 2[[\nu,\eta],\nu^*] + 2[[\nu^*,\eta],\nu] \\ &= 4[[\nu,\eta],\nu^*] - 2[[\nu,\nu^*],\eta] \\ &= 4[[\nu^*,\eta],\nu] + 2[[\nu,\nu^*],\eta].\end{aligned}$$

□

The same proof would also give that $\zeta_1 = 0$ if one knew that $\zeta_1$ is self-adjoint, but I know of no direct proof of this fact.

## 4.1 How to prove that $\xi = \mathrm{d}f(\nu)$

One is thus reduced to prove that $\xi = \mathrm{d}f_0(\nu)$ to complete the proof of Theorem 4.1. For this reason, from now on we always drop the 0 and write $f = f_0$ and $\rho = \rho_0$. We use here that for a given simple root $\alpha$, $f^*\omega_\alpha$ is an isomorphism between $T\Sigma$ and $f^*[\mathfrak{g}_\alpha]$, so that there exists a vector field $\nu$ on $\Sigma$ such that

$$\pi_\alpha(\xi) = f^*\omega_\alpha(\nu).$$

Define: $\hat{\xi} = \xi - \mathrm{d}f(\nu)$. We want to prove that $\hat{\xi} = 0$ or, equivalently, that $\mathrm{d}\hat{\xi} = 0$ or $\mathcal{J}(\hat{\xi}) = 0$; to do so, we give a list of axioms that $\xi$ satisfies and that turn out to be satisfied by $\hat{\xi}$, too.

**Definition 4.7.** The *cyclic Pfaffian system* is the set of differential forms on $\Sigma$:

$$\left(\omega_j\right)_{j=0}^4 := \left(f^*\omega_0, f^*\omega_1, f^*\phi \wedge f^*\phi, f^*\phi^* - f^*\phi^\dagger, \hat{\pi}_0(f^*\phi \wedge f^*\phi^\dagger)\right),$$

where $\hat{\pi}_0$ is the projection onto the orthogonal of $[\mathfrak{t}]$ in $[\mathfrak{h}]$. An *infinitesimal deformation of cyclic surfaces* is a section $\eta$ of $[\mathfrak{g}_0]$ (i.e. $\bar{\eta} = \eta$) such that, for every $j = 0, \ldots, 4$,

$$i_\eta \mathrm{d}^\nabla(\omega_j) = -\mathrm{d}^\nabla\left(i_\eta\omega_j\right) \tag{11}$$

(here, $i_\eta$ is the contraction by $\eta$, i.e. if $\phi$ is a $p$-form, then $i_\eta\phi$ is the $(p-1)$ form given by $i_\eta\phi(\cdot) = \phi(\eta, \cdot)$). It is called an infinitesimal deformation of *closed* cyclic surfaces if, furthermore, it is $\mathrm{Ad}(\rho)$-equivariant.

Clearly, the relations $\omega_j = 0$ are those that define $f$ to be cyclic, together with the condition of nowhere-vanishing of $f^*\omega_\alpha$. The following is true:

**Lemma 4.8.** *Both $\xi$ and $\hat{\xi}$ are infinitesimal deformations of cyclic surfaces.*

*Sketch of proof.* Infinitesimal deformations of (closed) cyclic surfaces form a vector space, and since $\mathrm{d}f(\nu)$ is trivially one such, it is enough to prove the Lemma for $\xi$. We already proved the "closed" part in Remark 4.2. To prove the rest, Labourie first reduces to replacing $(f^*[\mathfrak{g}], \nabla)$ by a trivial vector bundle with its flat connection; this is possible because the definition of an infinitesimal deformation of cyclic surfaces is local, so it suffices to consider a parallel local frame. Then, considering $f_t$ as a unique map $\Sigma \times (-1, 1) \to X$, the conclusion follows from the Lie-Cartan formula and some computations.

In alternative, a less elegant but more straightforward method is to remark via an explicit computation that for every $f^*[\mathfrak{g}]$-valued $p$-form $\alpha$ on $\Sigma \times (-1, 1)$,

$$\frac{\partial}{\partial t} f_t^* \alpha\Big|_{t=0} = f^*\Big(i_\xi \mathrm{d}^\nabla \alpha + \mathrm{d}^\nabla i_\xi \alpha\Big),$$

and since the left hand side is 0 for every $\alpha = \omega_j$, the conclusion follows, as well. $\square$

The advantage we have in dealing with $\hat{\xi}$ instead of $\xi$ directly is that for at least one simple root $\alpha$, $\hat{\xi}_\alpha = 0$. However, the proof of Corollary 4.6 does not extend automatically: We have used the non-infinitesimal Theorem 3.5, implying that cyclic maps project to harmonic maps in $G/K$. The infinitesimal harmonic equation (for $\xi$ self-adjoint) is $\mathcal{J}(\xi) = 0$, which, as we have seen, implies $\xi = 0$. Although he uses different notations, the first part of Labourie's proof is just a long set of computations proving explicitly that the definition of an infinitesimal deformation of cyclic maps $\eta$ implies $\mathcal{J}(\eta_0) = \mathcal{J}(\eta_1) = 0$ (he does not remark that $\mathcal{J}$ preserves the decomposition (7), so his notations are somewhat more lengthy). I avoid rewriting his computations, that in every situation may be re-done by hand using relations (11) and (4) to commute operators. Jumping to conclusions, he gets:

$$\mathcal{J}(\hat{\xi}_0) = 0 \text{ (Proposition 7.6.2)} \quad \text{and} \quad \mathcal{J}(\hat{\xi}_1) = 0 \text{ (Proposition 7.6.1)}.$$

It is left to prove that $\hat{\zeta} = \hat{\xi}_Z$ and $\hat{\zeta}^\dagger$ vanish. Here the above strategy should fail, because, without knowing a priori that $\dot{J}_0 = 0$, equation (10) would give an a priori non-zero expression for $\mathcal{J}(\hat{\zeta} + \hat{\zeta}^\dagger)$. Here is where Labourie uses the hypothesis $\hat{\xi}_\alpha = 0$. One also has to distinguish between the case $\mathrm{SL}_3(\mathbb{R})$ and the other ones, that turn out to be easier since one can exploit the relation

$$\pi_1(\hat{\zeta} \wedge \phi) = 0. \tag{12}$$

Equation (12) follows from the relation on the infinitesimal deformation of cyclic maps with respect to $f^*\omega_1$, that gives:

$$\partial\hat{\xi}_1 = 2\pi_1\big((\hat{\xi}_1 + \hat{\zeta}) \wedge \phi\big) \quad \text{(Proposition 7.4.2)},$$

and we have already mentioned that $\hat{\xi}_1 = 0$. This relation can be used, when $G^{\mathbb{C}} \neq \mathrm{SL}_3(\mathbb{C})$, to "propagate" $\hat{\xi}_\alpha = 0$ to all other simple roots: Indeed, if $\beta$ is another simple root such that $\alpha + \beta = \gamma$ is a root, then necessarily $x_\gamma \in [\mathfrak{g}_1]$, and actually the projection to $[\mathfrak{g}_\gamma]$ of (12) gives:

$$\hat{\xi}_\alpha \wedge \phi_\beta + \hat{\xi}_\beta \wedge \phi_\alpha = 0,$$

since $\gamma$ may be written uniquely as sum of simple roots. As $\hat{\xi}_\alpha = 0$ and $\phi_\alpha \neq 0$, also $\zeta_\beta = 0$, and since the Dynkin diagram is connected this procedure works for all the simple roots. The same trick can be exploited to prove that $\hat{\xi}_\eta = 0$, since if $\eta - \alpha$ is a root, then $x_{\eta-\alpha} \in [\mathfrak{g}_1]$, as well.

We are only left to deal with $G_0 \cong \mathrm{SL}_3(\mathbb{R})$. This is done in a more explicit way, that we now only sketch. There are but three positive roots, $\alpha$, $\beta$ and $\eta$, and we know that $\hat{\xi}_\alpha = \hat{\xi}_{-\alpha} = 0$ (since $\hat{\xi}$ is real and $\lambda$ exchanges $[\mathfrak{g}_\alpha]$ and $[\mathfrak{g}_{-\alpha}]$). Thus we can write, for example

$$\hat{\zeta} = \mu_\eta x_\eta + \mu_{-\beta} x_{-\beta}, \quad \phi^\dagger = \psi_{-\eta} x_{-\eta} + \psi_\alpha x_\alpha + \psi_\beta x_\beta$$

(as $\phi^* = \phi^\dagger$, the coefficients of $\phi$ will just be the conjugates of the $\psi$'s). It is clear that

$$[\hat{\zeta}, \phi^\dagger] = -\mu_{-\beta}\psi_\beta [x_\beta, x_{-\beta}] + \mu_\eta \psi_{-\eta}[x_\eta, x_{-\eta}].$$

We have seen that $[x_\eta, x_{-\eta}] = h_\eta \in \mathfrak{t}_{\mathbb{C}}$, and in the same way as (12), one has in this case $\hat{\pi}_0(\hat{\zeta} \wedge \phi^\dagger) = 0$. As a consequence, since $\psi_\beta$ never vanishes, $\mu_{-\beta} = 0$, too. A similarly explicit analysis, this time using an expression for $\nabla\hat{\zeta} \wedge \phi$, allows to deduce that $\hat{\xi}_\eta = 0$, too.

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