Introduction to Higgs Bundles

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1 Introduction

Let X be a compact Kahler manifold. We can think of classes in $H^1(X, \mathbb{C})$ as homomorphisms of $\pi_1(X)$ into $\mathbb{C}$. On the other hand, the Hodge theorem gives a decomposition

$$H^1(X, C) = H^{0,1}(X) \oplus H^{1,0}(X) = H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X).$$

Therefore, homomorphisms $\pi_1(X) \to \mathbb{C}$ are the same as an element of $H^1(X, \mathcal{O}_X)$, i.e. a holomorphic line bundle of degree 0, and an element of $H^0(X, \Omega^1_X)$, i.e. a holomorphic 1-form. In these notes, we describe an analogous correspondence for the case of represenations into a nonabelian Lie Group G, focusing in particular on the case $G = GL(n, \mathbb{C})$ [5]. Theorems are given without proof, and the reader may consult the references for further information. A nice overview of the following with proofs can be found in [7].

1.1 Preliminary Definitions

First we give a review of complex vector bundles. Many outstanding sources on this subject exist. For these notes, the author followed Wells' *Differential Analysis on Cmoplex Manifolds* [6].

Let X be a complex manifold. Informally, a smooth complex vector bundle on X is a family of smoothly varying complex vector spaces over X. Formally, it is a smooth manifold E and a projection $\pi : E \to X$ satisfying $\forall x \in X$

1. $\pi^{-1}(x)$ has the structure of a complex vector space

2. there is an open neighborhood $U \subset X$ of x and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ satisyting $\pi \circ \phi(E_x) = x$ and $\phi|_{E_x}$ composed with projection onto $\mathbb{C}^k$ is a vector space isomorphism.

2. gives us an open covering $\{U_\alpha\}$ of M and "local trivializations" $\phi_\alpha : \pi^{-1} \to U_\alpha \times \mathbb{C}^k$. From this we define the "transition functions" $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{C})$. The data of an open covering $\{U_\alpha\}$ and transition functions $\{g_{\alpha\beta}\}$ satisying the cocycle condition $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ is exactly what is required to define a vector bundle.

A smooth section of a vector bundle is a collection of smooth functions s_α defined on U_α satisfying $s_\alpha = g_{\alpha\beta}s_\beta$.The space of all sections of E is $\Gamma(X, E)$. There is no canonical way to differentiate sections along vector fields in M. To do this, we define a connection on E to be a map $\nabla : \Gamma(X, E) \to \Gamma(X, E \otimes \Omega^0(X))$ satisfying the Leibniz condition

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

for f a smooth function and $s \in \Gamma(X, E)$. A del-bar operator is a map $\bar{\partial}_E : \Gamma(X, E) \to \Gamma(X, E \otimes \Omega^{0,1}(X)$ satisfying

$$\bar{\partial}_E(fs) = \bar{\partial}(f) \otimes s + f\bar{\partial}_E(s)$$

for a smooth function f and $s \in \Gamma(X, E)$. A del-bar operator is integrable if $\bar{\partial}_E^2 = 0$

An hermitian metric h is the assignment of an hermitian inner product h_x to each fiber E_x of E. This assignment is required to be soothly varying in the sense that for two local sections s_1, s_2, the function $\langle s_1, s_2\rangle_h(x) = \langle s_1(x), s_2(x)\rangle_h$ is smooth. A connection ∇ is compatible with h if it satisfies

$$d\langle s_1, s_2\rangle_h = \langle \nabla(s_1), s_2\rangle_h + \langle s_1, \nabla(s_2)\rangle_h .$$

A smooth vector bundle E is holomorphic if the transition functions defining it are holomorphic. In this case we can find local holomorphic frames $\{s_1, \ldots, s_k\}$. In this frame, we can locally define a del-bar operator $\bar{\partial}_E$ by

$$\bar{\partial}_E(fs_i) = \bar{\partial}(f) \otimes s.$$

Actually, this globally defines an integrable del-bar operator. Conversely, a slight modification of the proof of the Newlander-Nirenberg theorem shows an integrable del-bar operator gives a holomorpic structure on E. Finally, a connection is compatible with a given holomorphic structure if the $(0,1)$ part of the connection is the same as a del-bar operator.

Given a holomorphic vector bundle E and an hermitian metric h, there is a unique connection, called the Chern connection, which is compatible with h and the holomorphic structure. We will denote this connection by d_A. In a holomorphic frame, the connection 1-form is $\theta = h^{-1}\partial h$. The curvature of the Chern connection will be denoted $F_{(\bar{\partial}_E,h)}$.

The curvature of a connection ∇ is $F_\nabla = \nabla^2 : \Gamma(X,E) \to \Gamma(X, E \otimes \Omega^2(X)$. In fact, F_∇ is $C^\infty(X)$ linear, i.e. a setion of $End(E) \otimes \Omega^2(X)$. A connection is flat if $F_\nabla = 0$. Equivalently, a connection is flat if there exist local parallel frames, i.e. local frames $\{s_1,\ldots,s_k\}$ satisfying $\nabla(s_i) = 0$.

Now let X be a compact Riemann surface. Holomorphic line bundles on X are identified with elements of the sheaf comhomology group $H^1(X, \mathcal{O}^*)$. The degree of a holomorphic line bundle $\mathcal{L}$ is defined to be the image of $\mathcal{L}$ under the map

$$H^1(X,\mathcal{O}^*) \to H^2(X,\mathbb{Z})$$

where $H^2(X,\mathbb{Z})$ is canonically identified with $\mathbb{Z}$. The degree of a general holomorphic vector bundle E is defined to be the degree of the line bundle $det(E)$. If we fix any hermitian metric h, then by the Chern-Weil theory we can compute the degree with the formula

$$deg(E) = \frac{\sqrt{-1}}{2\pi}\int trF_{(\bar{\partial}_E,h)}.$$

Finally, we define the slope of a vector bundle to be the ratio $\mu(E) = deg(E)/rk(E)$.

2 The Objects

We will restrict ourselves to the case where X is a compact Riemann surface of genus $g \geq 2$. Much of the below is still true with slight modification in the case of compact Kahler manifolds, though we will not consider this case here.

2.1 Representations of π_1 and Flat Bundles

Let $G = GL(n,\mathbb{C})$. Recall that $\pi_1(X) =< a_1, b_1, \ldots, a_g, b_g | \Pi[a_i,b_i] = 1 >$. A representation of $\pi_1(X)$ is a homomorphism $\rho : \pi_1(X) \to GL(n,\mathbb{C})$. Given a representation ρ of $\pi_1(X)$, we can construct a flat bundle by $E = \tilde{X} \times_\rho \mathbb{C}^n = \tilde{X} \times \mathbb{C}^n/$ where $\tilde{X}$ is the simply-connected cover of X and $\pi_1(X)$ acts on $\tilde{X}$ by deck transformations and on $\mathbb{C}^n$ by ρ. Conversely, since the holonomy of a flat connection depends only on the homotopy class of a path, a flat connection gives a representation of $\pi_1(X)$.

2.2 Higgs Bundles

A Higgs bundle is a holomorphic bundle $(E, \bar{\partial}_E)$ and a holomorphic section ϕ of $End(E) \otimes \mathcal{K}_X$. A subbundle F of E is ϕ-invariant if $\phi(F) \subset F \otimes \mathcal{K}_X$. A Higgs bundle (E, ϕ) is semistable if for all ϕ-invariant subbundles F, we have $\mu(F) \leq \mu(E)$. (E, ϕ) is stable if the inequality is strict for all F. (E, ϕ) is polystable if it is a direct sum of stable Higgs bundles of the same slope.

Here is an example of a stable Higgs Bundle which will be of importance later. Let $K_X^{1/2}$ be a square root of the canonical bundle of X, and define $E = K_X^{1/2} \oplus K_X^{-1/2}$. Define the higgs field ϕ by

$$\phi = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

where 1 is the identity section of the line bundle $Hom(\mathcal{K}_X^{1/2}, \mathcal{K}_X^{-1/2} \otimes \mathcal{K}_X) = \mathcal{O}$. The only invariant line subbundle is $\mathcal{K}_X^{-1/2}$, which has slope $1 - g$. $\mu(E) = 0$, so we conclude that (E, ϕ) is stable.

Define $\mathcal{M}_{Dol}^0$ to be the moduli space of polystable higgs bundles of degree 0 (sometimes referred to as the Dolbeault moduli space). We conclude this section with the definition of the Hitchin fibration $\mathcal{M}_{Dol}^0 \to \bigoplus_i H^0(X, \mathcal{K}^i)$. This is given by sending the pair (E, Φ) to the coefficients of λ^i in the characteristic polynomial $det(\lambda I - \Phi)$ of the Higgs field Φ.

3 The Correspondence

We now wish to describe the relationship between higgs bundles and flat bundles.

3.1 From Higgs Bundles to Flat Bundles

Let (E, ϕ) be a Higgs bundle. If we fix an hermitian metric h on E, then $\nabla = d_A + \phi + \phi^*$ defines a connection on E. This is our candidate flat connection.

However, its curvature $F_\nabla = F_A + [\phi, \phi^*]$ may not vanish. Thus our higgs bundle corresponds to a flat bundle precisely when Hitchin's Equations $F_A + [\phi, \phi^*] = 0$ are satisfied. We will also be interested in bundles on nonzero degree, for which the equations become

$$\sqrt{-1}\Lambda(F_A + [\phi, \phi^*]) = \mu(E). \tag{1}$$

We wish then to find higgs bundles satisfying these equations. It turns out, the obstruction to doing this is exactly the stability condition for higgs bundles defined previously, as evidenced by the following proposition:

Proposition: If (E, ϕ) is a higgs bundle satisfying (1), then (E, ϕ) is polystable.

However, the converse of this also true:

Theorem: (Hitchin [4], Simpson[5]) If (E, ϕ) is polystable, then it admits a metric satisfying (1).

3.2 From Flat Bundles to Higgs Bundles

Let (E, ∇) be a flat bundle. If we again fix an hermitian metric h, then we can define Ψ, the obstruction to ∇ being untiary. In a local frame $\{s_i\}$, Ψ is defined by the equation

$$\langle \Psi s_i, s_j \rangle = \langle \nabla s_i, s_j \rangle + \langle s_i, \nabla s_j \rangle - d \langle s_i, s_j \rangle .$$

Ψ is hermitian, so we can write $\Psi = \phi + \phi^*$ for some 1-form valued endomorphism ϕ. However, ϕ may not be holomorphic. In fact, ϕ is holomorphic iff $d_A^* \Psi = 0$ (we also require $d_A \Psi = 0$, which we get for free since $\nabla = d_A + \Psi$ is assumed to be flat). This is proven by a computation using the Kahler identies for the metric connection d_A. The equation $d_A^* \Psi = 0$ is related to the theory of harmonic maps.

We can view an hermitian metric h as a ρ-equivariant map $u : \tilde{X} \to GL(n, \mathbb{C})/U(n)$. Define the energy density of u to be

$$E(u) = \frac{1}{2} \int_M |du|^2 \omega.$$

The hermitian metric corresponding to u is said to be harmonic if u is a critical point of E (i.e. if u is a harmonic map). First we note the following lemma:

Lemma: $E(u) = 2||\Psi||^2$.

From this it is easy to compute the Euler-Lagrange equations $d_A^* \Psi = 0$. Thus we conclude that ϕ defined above is holomorphic if and only if h is a harmonic metric, and the problem of relating flat bundles to higgs bundles becomes one of finding harmonic metrics.

We finish with the following theorem:

Theorem: (Corlette, Donaldson) (E, ∇) admits a harmonic metric iff ∇ is semisimple [1][2].

3.3 Putting Everything Together: The Nonabelian Hodge Correspondence

Recall $\mathcal{M}_{Dol}^0$ isthe moduli space of polystable higgs bundles of degree 0. Define $\mathcal{M}_B = Hom(\pi_1(X), G)/G$ to be the character variety (or Betti moduli space). In sections 3.1 and 3.2, we defined maps from each moduli space to the other, which are clearly inverse to one another. Thus we have the following theorem:

Nonabelian Hodge Theorem The above defines a homeomorphism $\mathcal{M}_B \simeq \mathcal{M}_{Dol}^0$.

3.4 A Remark about $G = SL(n, \mathbb{C})$

In the above, we only considered the case $G = GL(n, \mathbb{C})$. The case $G = SL(n, \mathbb{C})$ follows similarly, requiring that $det(E)$ be fixed and that ϕ be traceless on the higgs bundle side.

4 The Hithchin Component

Now we wish to identify a special component in the space of $SL(n, \mathbb{R})$ representations, called the Hitchin component. First, we must identify the real representations in the $SL(N, \mathbb{C})$ representation variety. To do this, we must know the relationship between the higgs bundles corresponding to a represention $\rho : \pi_1 \to SL(n, \mathbb{C})$ and the higgs bundle corresponding to the conjugate representation $\bar{\rho}$.

Proposition: If the higgs bundle (E, ϕ) corresponds to a representation

$\rho \in Hom(\pi_1(X), SL(n\mathbb{C}))/SL(n,\mathbb{C})$, then the higgs bundle (E^*, ϕ^t) corresponds to $\bar{\rho}$ [3][5].

Thus the higgs bundles corresponding to real representations are exactly those fixed by the above involution. We have already seen the following examples:

$$E = \mathcal{K}^{1/2} \oplus \mathcal{K}^{-1/2}$$
$$\Phi = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}$$

where $\alpha \in H^0(X, \mathcal{K}^2)$ is a holomorphic quadratic differential. In fact, these examples represent an entire component of the character variety $Hom(\pi_1(X), SL(2,\mathbb{R}))//SL(2,\mathbb{R})$.

Note that there is a unique irreducible representation $SL(2,\mathbb{R}) \hookrightarrow SL(n,\mathbb{R})$ given by the (n-1)-th symmetric power of the standard representation. The Hitchin component Hit_n is then defined to be the path component of $Hom(\pi_1(X), SL(n,\mathbb{R}))//SL(n,\mathbb{R})$ containing the image of the above 2-dimensional representations after post-composing with this irreducible representation.

On the higgs bundle side, we have

$$\tilde{E} = S^{n-1}E = \mathcal{K}^{(n-1)/2} \oplus \mathcal{K}^{(n-3)/2} \oplus \ldots \oplus \mathcal{K}^{-(n-3)/2} \oplus \mathcal{K}^{-(n-1)/2}$$
$$S^{n-1}\Phi = \begin{bmatrix} 0 & \alpha & & & 0 \\ 1 & 0 & \alpha & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & \alpha \\ 0 & & & 1 & 0 \end{bmatrix}$$

We have some additional freedom in defining the higgs field $\tilde{\Phi}$. Let $\alpha_i \ni H^0(X, \mathcal{K}^i)$ for $2 \leq i \leq n$. Then we can define

$$\tilde{\Phi} = \begin{bmatrix} 0 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ 1 & 0 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \alpha_3 \\ & & 1 & 0 & \alpha_2 \\ 0 & & & 1 & 0 \end{bmatrix}.$$

Choosing an appropriate basis of conjugation invariant polynomials, $(\tilde{E}, \tilde{\Phi})$ gives a section of the Hitchin Fibration $\mathcal{M}^0_{Dol} \to \bigoplus_{i=2}^n H^0(X, \mathcal{K}^i)$. A dimension count (Riemann-Roch gives $3g - 3$ as the complex dimension of $\bigoplus_{i=2}^n H^0(X, \mathcal{K}^i)$) and application of the inverse function theorem (here noting that the image of this section is in the smooth locus of $\mathcal{M}^)_{Dol}$) proves we have constructed a full path component of $\mathcal{M}^0_D ol$. Hence we have shown the character variety $Hom(\pi_1(X), SL(n, \mathbb{R}))//SL(n, \mathbb{R})$ contains a path component homeomorphic to $\mathbb{R}^{6g-6}$.

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