EXERCISES FOR HIGGS BUNDLE COURSE

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1) Let $\widetilde{S}$ be the universal cover of S and let $p : \widetilde{S} \to S$ be the covering map. Given a group homomorphism $\rho : \pi_1(S) \to G$, show that $\pi_1(S)$ acts on $\widetilde{S} \times G$ and that the quotient by this action (denoted $\widetilde{S} \times_\rho G$) can be viewed as the total space of a $G-$bundle over S.

Show that the bundle has local trivializations over an atlas of open sets $\{U_\alpha \subset S\}$ for which the transition functions on non-empty intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$ are *locally constant* maps $g_{\alpha\beta} : U_{\alpha\beta} \to G$. That is, show that $\widetilde{S} \times_\rho G$ defines a flat bundle (or local system).

Solution: Let $s_0 \in S$ and select $\gamma \in \pi_1(S, s_0)$. Define the action of $\pi_1(S, s_0)$ on $\widetilde{S} \times G$ via the formula:

$$\gamma \cdot (\widetilde{s}, g) = (\gamma \cdot \widetilde{s}, \rho(\gamma)g).$$

This action is free and properly discontinuous since the action of $\pi_1(S, s_0)$ on $\widetilde{S}$ is free and properly discontinuous. Hence, $\widetilde{S} \times_\rho G$ is a smooth manifold and the map

$$\widetilde{S} \times G \to \widetilde{S} \times_\rho G$$

is a smooth submersion. We must show that there is a smooth, proper, free action of G on the right of $\widetilde{S} \times_\rho G$ such that the quotient by this action is diffeomorphic to S.

Observe that given $g' \in G$, a right $G-$action on $\widetilde{S} \times G$ is defined by

$$(\widetilde{s}, g) \cdot g' = (\widetilde{s}, gg').$$

This action descends to a smooth right $G-$action on $\widetilde{S} \times_\rho G$ by virtue of the fact that the projection map

$$\widetilde{S} \times G \to \widetilde{S} \times_\rho G$$

is a smooth map. Furthermore, this right action of G on $\widetilde{S} \times_\rho G$ is proper since the right action of G on itself has this property. Now, define a map

$$\pi : \widetilde{S} \times_\rho G \to S$$

via the formula $\pi(\widetilde{s}, g) = p(\widetilde{s})$. It is immediate that this map is well defined. Next,

$$\pi(\widetilde{s}, g) = \pi(\widetilde{t}, h)$$

if and only if

$$\widetilde{t} = \gamma \cdot \widetilde{s}$$

for some $\gamma \in \pi_1(S, s_0)$.

Observe that,

$$(\widetilde{t}, h) \cdot h^{-1}\rho(\gamma)g = (\gamma \cdot \widetilde{s}, \rho(\gamma)g) = (\widetilde{s}, g)$$

in $\widetilde{S} \times_\rho G$ and so G acts transitively on each fiber of the map π. G also acts freely on the fiber since the right action of G on itself is free. Lastly,

$$(\gamma \cdot \widetilde{s}, g) \cdot g^{-1}\rho(\gamma)g = (\gamma \cdot \widetilde{s}, \rho(\gamma)g) = (\widetilde{s}, g)$$

which proves that the quotient of $\widetilde{S} \times_\rho G$ by the right action of G is exactly S and the projection map

$$\widetilde{S} \times_\rho G \to \widetilde{S} \times_\rho G/G = S$$

coincides with the map π defined above. This completes the proof that $\widetilde{S} \times_\rho G$ is the total space of a principal G-bundle over S.

We will now construct the flat structure on this bundle. Choose a trivializing atlas for the universal covering given by a good open cover U_α (i.e. each U_α simply connected) and local sections $s_\alpha : U_\alpha \to \widetilde{S}$. If $U_\alpha \cap U_\beta \neq \emptyset$ and connected, then $s_\alpha = \gamma_{\alpha\beta} s_\beta$ for some $\gamma_{\alpha\beta} \in \pi_1(S, s_0)$. This allows us to define local trivializations,

$$\Psi_\alpha : U_\alpha \times G \to \widetilde{S} \times_\rho G|_{U_\alpha}$$

by the formula

$$\Psi_\alpha(x, g) = (s_\alpha(x), g).$$

If $x \in U_\alpha \cap U_\beta$, then we have the competing local trivialization

$$\Psi_\beta(x, g) = (s_\beta(x), g).$$

But, in the bundle $\widetilde{S} \times_\rho G$ we have the identities

$$(s_\alpha(x), g) = (\gamma_{\alpha\beta} s_\beta(x), g) = (s_\beta(x), \rho(\gamma_{\alpha\beta})^{-1} g).$$

Thus the transition functions $\Psi_\alpha^{-1} \circ \Psi_\beta$ take the form

$$\Psi_\alpha^{-1} \circ \Psi_\beta(x, g) = (x, \rho(\gamma_{\alpha\beta})g).$$

This is independent of the $x \in U_\alpha \cap U_\beta$ where we assumed $U_\alpha \cap U_\beta$ is connected. Thus, the transition maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$

defined by $g_{\alpha\beta}(x) = \rho(\gamma_{\alpha\beta})$ define a flat structure on $\widetilde{S} \times_\rho G$.

10) Let $V \to X$ be a vector bundle given by local trivializations

$$\Psi_\alpha : U_\alpha \times \mathbb{R}^n \to V|_{U_\alpha}$$

and transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R}).$$

Let $s : X \to V$ be a global section of V defined by the local sections

$$s_\alpha : U_\alpha \to U_\alpha \times \mathbb{R}^n.$$

Show that in general the locally defined $\mathbb{R}^n$-valued 1-forms $\{ds_\alpha\}$ do not defined a global section of $V \otimes T^*X$, but that they do so if the transition functions are locally constant.

Solution: Write each local section as an n-tuple of smooth functions

$$s_\alpha(x) = (x, s_\alpha^1(x), ..., s_\alpha^n(x)).$$

Then we have the identity

$$s_\alpha^i(x) = g_{\alpha\beta j}^i(x) s_\beta^j(x)$$

where the right hand side is summed over the repeated index j. Applying the d operator and using the Leibniz rule we obtain,

$$d(s_\alpha^i) = d(g_{\alpha\beta j}^i) s_\beta^j + g_{\alpha\beta j}^i d(s_\beta^j).$$

Provided the transition functions are locally constant, the first summand on the left hand side vanishes leaving

$$d(s_\alpha^i) = g_{\alpha\beta j}^i d(s_\beta^j).$$

This is precisely the transformation rule for a global section of $V \otimes T^*X$. This locally defined operator extends to give a globally defined operator

$$\nabla : \Gamma(V) \to \Gamma(V \otimes T^*X)$$

which defines a covariant derivative on the bundle V. The fact that $d \circ d = 0$ translates into the fact that the curvature of this connection ∇ is identically zero. Hence, we see that a flat bundle defined via locally constant transition functions gives rise to a vector bundle with a flat connection.