Introduction to Higgs bundles
Lecture III

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Disclaimer

These slides are precisely as they were during the lecture on July 27, 2012. As such, they contain several omissions and inaccuracies, in both the mathematics and the attributions. Some of these, it must be admitted, are blemishes which reflect the author's limitations, but others reflect the fact that:

- The slides formed but one part of the lectures. They were accompanied by verbal commentary designed to explain and embellish the contents of the slides
- This is not a paper. Any talk has to strike a balance between accuracy and accessibility. This balance inevitably involves the inclusion of some half-truths and/or white lies.

The author apologizes to anyone who is in any way led astray by the inaccuracies or slighted by the omissions.

The plan for this lecture

- Features of the Higgs bundle moduli spaces
- An example

Synopsis of Lecture II

A **Higgs bundle** on a Riemann surface $\Sigma = (S, J)$ is a pair $(\mathcal{E}, \varphi)$ where

- $\mathcal{E} = (E, \overline{\partial}_E)$ is a rank n holomorphic bundle,
- $\varphi \in \Omega^{(1,0)}(End(\mathcal{E}))$ is holomorphic (i.e. $\overline{\partial}_E \varphi = 0$)

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$$\mathcal{M}_{Higgs}(\Sigma, \mathrm{GL}(n, \mathbb{C})) = \left\{ \begin{array}{c} \text{Isomorphism classes of polystable, rank n,} \\ \text{degree zero Higgs bundles on } \Sigma \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \text{Isomorphism classes of } (\mathcal{E}, \varphi) \text{ where } \mathcal{E} \\ \text{admits a metric satisfying Hitchin's equation} \end{array} \right\}$$

Properties of $\mathcal{M}_{Higgs}$

$$\begin{array}{c} \mathrm{genus}(\Sigma) = g \\ \mathrm{rank}(E) = n \end{array}$$

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- defines an action on $\mathcal{M}_{Higgs}$ by

$$\lambda[\mathcal{E}, \varphi] = [\mathcal{E}, \lambda\varphi]$$

The Hitchin function on $\mathcal{M}_{Higgs}$

$$
\begin{aligned}
f : \mathcal{M}_{Higgs} &\to \mathbb{R} \\
[\mathcal{E}, \varphi] &\mapsto \int_\Sigma |\varphi|_H^2 dvol = ||\varphi||_H^2
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- critical points of f = fixed points of the $\mathbb{C}^*$-action [Frenkel]
- Useful as Morse function! ($\mathrm{GL}(2,\mathbb{C})$ [Hitchin]; $\mathrm{GL}(3,\mathbb{C})$ [Gothen]..)

The Hitchin fibration

- In local frames over $\{U_\alpha\}$, $\varphi = \phi_\alpha dz$ with
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- generic fibers are tori
- $\dim_C \bigoplus_{i=1}^{N} H^0(K^{d_i}) = \frac{1}{2}\dim_{\mathbb{C}} \mathcal{M}_{Higgs}$
- this defines the Hitchin integrable system

[Hitchin]

- $E = (\coprod_\alpha U_\alpha \times \mathbb{C}^2)/\{g_{\alpha\beta}\}$; $\overline{\partial} g_{\alpha\beta} = 0$ for holomorphic frames
- $\varphi = \{\varphi_\alpha = \phi_\alpha dz\}$ with $\overline{\partial}\phi_\alpha = 0$

Rank 2 $\implies$ the only subbundles of $\mathcal{E}$ are line subbundles!

- $\mathcal{L} \subset \mathcal{E}$ means $g_{\alpha\beta} = \begin{bmatrix} l_{\alpha\beta} & * \\ 0 & q_{\alpha\beta} \end{bmatrix}$ where $\{l_{\alpha\beta}\}$ define $\mathcal{L}$.

Stability for $(\mathcal{E}, \varphi)$

Two cases:

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2. there exists an $\mathcal{L}$ such that $\deg(\mathcal{L}) > 0$

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Case 1: $\mathcal{E}$ is (semi)stable and $(\mathcal{E}, \varphi)$ is (semi)stable for all φ.

Case 2: The 'bad' $\mathcal{L}$ is unique and $(\mathcal{E}, \varphi)$ is (semi)stable if and only if $\mathcal{L}$ is not φ-invariant.

Example ($\mathcal{E} = \mathcal{L} \oplus \mathcal{Q}$ with $\deg(\mathcal{L}) > 0$)

- $\varphi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies \boxed{\varphi(\mathcal{L}) \subset \mathcal{L}K_\Sigma \iff c = 0}$
- $(\mathcal{E}, \varphi)$ is stable if and only if $\boxed{c \neq 0}$

A special case

- $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^*$ with $\deg(\mathcal{L}) > 0$
- $\varphi = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ with $c \neq 0$

Note:
If $\mathcal{L}$ is defined by $\{l_{\alpha\beta}\}$ then $\mathcal{L}^*$ is defined by $\{l_{\alpha\beta}^{-1}\}$.
This explains why $\mathcal{L}^* = \mathcal{L}^{-1}$.

Lecture continued on the blackboard...