Introduction to Higgs bundles Lecture III

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These slides are precisely as they were during the lecture on July 27, 2012. As such, they contain several omissions and inaccuracies, in both the mathematics and the attributions. Some of these, it must be admitted, are blemishes which reflect the author's limitations, but others reflect the fact that:

- The slides formed but one part of the lectures. They were accompanied by verbal commentary designed to explain and embellish the contents of the slides
- This is not a paper. Any talk has to strike a balance between accuracy and accessibility. This balance inevitably involves the inclusion of some half-truths and/or white lies.

The author apologizes to anyone who is in any way led astray by the inaccuracies or slighted by the omissions.

- Features of the Higgs bundle moduli spaces
- An example

Synopsis of Lecture II

A **Higgs bundle** on a Riemann surface $\Sigma = (S, J)$ is a pair (\mathcal{E}, φ) where • $\mathcal{E} = (E, \overline{\partial}_E)$ is a rank n holomorphic bundle, • $\varphi \in \Omega^{(1,0)}(End(\mathcal{E}))$ is holomorphic (i.e. $\overline{\partial}_E \varphi = 0$)

$$\operatorname{\mathsf{Rep}}^{red}(\pi_1(S),\operatorname{GL}(n,\mathbb{C})) \leftrightarrow \mathcal{M}_{Higgs}(\Sigma,\operatorname{GL}(n,\mathbb{C}))$$

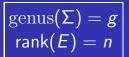
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A Higgs bundle on a Riemann surface Σ = (S, J) is a pair (E, φ) where
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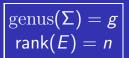
$$\mathcal{M}_{Higgs}(\Sigma, \operatorname{GL}(n, \mathbb{C})) = \begin{cases} \operatorname{Isomorphism \ classes \ of \ polystable, \ rank \ n,} \\ \operatorname{degree \ zero \ Higgs \ bundles \ on \ } \end{cases} \\ = \begin{cases} \operatorname{Isomorphism \ classes \ of \ } (\mathcal{E}, \varphi) \ \text{where \ } \mathcal{E} \\ \operatorname{admits \ a \ metric \ satisfying \ Hitchin's \ equation} \end{cases}$$

Properties of \mathcal{M}_{Higgs}



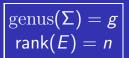
• complex analytic with dim_{\mathbb{C}} $\mathcal{M}_{Higgs} = 2n^2(g-1) + 2$

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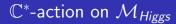


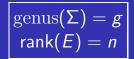
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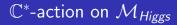
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- hyperkahler (on smooth locus)

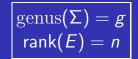




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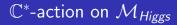
 $\begin{array}{c} \varphi(\mathcal{E}') \subset \mathcal{E}' \otimes \mathsf{K}_{\Sigma} \\ \Longrightarrow (\lambda\varphi)(\mathcal{E}') \subset \mathcal{E}' \otimes \mathsf{K}_{\Sigma} \end{array}$

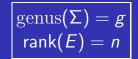
$$egin{array}{c} {
m Stability} \ {
m deg}(\mathcal{E}') < 0 \ {
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Higgs bundles

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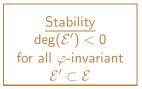




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 \bullet defines an action on $\mathcal{M}_{\textit{Higgs}}$ by

$$\lambda[\mathcal{E},\varphi] = [\mathcal{E},\lambda\varphi]$$

The Hitchin function on \mathcal{M}_{Higgs}

$$f: \mathcal{M}_{\mathsf{Higgs}} o \mathbb{R}$$

 $[\mathcal{E}, \varphi] \mapsto \int_{\Sigma} |\varphi|_{H}^{2} d extsf{vol} = ||\varphi||_{H}^{2}$

- Introduced by Hitchin in his original papers
- *f* is proper and bounded below (by zero)

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- Useful as Morse function! (GL(2, \mathbb{C}) [Hitchin]; GL(3, \mathbb{C}) [Gothen]..)

• In local frames over $\{U_{\alpha}\}$, $\varphi = \phi_{\alpha} dz$ with $\phi_{\alpha} \in \mathfrak{gl}(n, \mathbb{C})$ and $\phi_{\beta} = g_{\beta\alpha} \phi_{\alpha} g_{\alpha\beta}^{-1}$

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- generic fibers are tori
- dim_C $\bigoplus_{i=1}^{N} H^{0}(K^{d_{i}}) = \frac{1}{2} \dim_{\mathbb{C}} \mathcal{M}_{Higgs}$
- this defines the Hitchin integrable system

Hitchin

$$(\mathcal{E}, \varphi); \ \deg(E) = 0$$

Rank 2 \implies the only subbundles of \mathcal{E} are line subbundles!

•
$$\mathcal{L} \subset \mathcal{E}$$
 means $g_{\alpha\beta} = \begin{bmatrix} I_{\alpha\beta} & * \\ 0 & q_{\alpha\beta} \end{bmatrix}$ where $\{I_{\alpha\beta}\}$ define \mathcal{L} .

Stability for (\mathcal{E}, φ)

Two cases:

- deg $(\mathcal{L}) \leq 0$ for all $\mathcal{L} \subset \mathcal{E}$, or
- 2 there exists an \mathcal{L} such that $\deg(\mathcal{L}) > 0$

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Case 1: \mathcal{E} is (semi)stable and (\mathcal{E}, φ) is (semi)stable for all φ .

Case 2: The 'bad' \mathcal{L} is unique and (\mathcal{E}, φ) is (semi)stable if and only if \mathcal{L} is not φ -invariant.

Example
$$(\mathcal{E} = \mathcal{L} \oplus \mathcal{Q} \text{ with } \deg(\mathcal{L}) > 0)$$

• $\varphi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies \boxed{\varphi(\mathcal{L}) \subset \mathcal{L}K_{\Sigma} \iff c = 0}$
• (\mathcal{E}, φ) is stable if and only if $\boxed{c \neq 0}$

•
$$\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^*$$
 with deg $(\mathcal{L}) > 0$
• $\varphi = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ with $c \neq 0$

Note:

If \mathcal{L} is defined by $\{I_{\alpha\beta}\}$ then \mathcal{L}^* is defined by $\{I_{\alpha\beta}^{-1}\}$. This explains why $\mathcal{L}^* = \mathcal{L}^{-1}$.

Lecture continued on the blackboard...