Introduction to Higgs bundles
Lecture II

Steve Bradlow

Department of Mathematics
University of Illinois at Urbana-Champaign

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Disclaimer

These slides are precisely as they were during the lecture on July 25, 2012. As such, they contain several omissions and inaccuracies, in both the mathematics and the attributions. Some of these, it must be admitted, are blemishes which reflect the author's limitations, but others reflect the fact that:

- The slides formed but one part of the lectures. They were accompanied by verbal commentary designed to explain and embellish the contents of the slides
- This is not a paper. Any talk has to strike a balance between accuracy and accessibility. This balance inevitably involves the inclusion of some half-truths and/or white lies.

The author apologizes to anyone who is in any way led astray by the inaccuracies or slighted by the omissions.

Goals and plan for this mini-course

- What are Higgs bundles?
- How do they relate to surface group representations?
- What do we gain by taking the Higgs bundle point of view?

The Plan:

1. (Lectures I and II)Description of surface group representations from a bundle perspective, with necessary background to define Higgs bundles and to see their relation to the representations
2. (Lecture III) Examples and properties of Higgs bundles

Synopsis of Lecture I

- S a closed surface of genus g
- $\boxed{\rho : \pi_1(S) \to \mathrm{GL}(n, \mathbb{C})}$ $\longleftrightarrow$ $\boxed{E \to S \text{ with } F_D = 0}$
- Introduce:
  - $\Sigma = (S, J)$
  - $\mathcal{E} = (E, \overline{\partial}_E)$

Have: $\overline{\partial}_E : \Omega^0(E) \to \Omega^{(0,1)}(E)$ with $\overline{\partial}_E^2 = 0$

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- S a closed surface of genus g
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- Introduce:
  - $\Sigma = (S, J)$
  - $\mathcal{E} = (E, \overline{\partial}_E)$

$$\begin{array}{lcl} \text{Have:} & \overline{\partial}_E : \Omega^0(E) \to \Omega^{(0,1)}(E) & \text{with } \overline{\partial}_E^2 = 0 \\ \text{Seek:} & D : \Omega^0(E) \to \Omega^1(E) & \text{with } D^2 = 0 \end{array}$$

Hermitian metrics on complex bundles $\boxed{E \longrightarrow (M, J)}$

- A smoothly varying family $H(\ ,\ )$of hermitian metrics on E_x,
- defines $H(S, S')(x) \in \mathbb{C}$ for sections $S, S' \in \Omega^0(E)$
- Facilitates local **unitary** frames, and thus
- Local trivializations for which all $g_{\alpha\beta} \in \mathrm{U}(n) \subset \mathrm{GL}(n, \mathbb{C})$, i.e.
- Defines a reduction of structure group from $\mathrm{GL}(n, \mathbb{C})$ to $\mathrm{U}(n)$.

End of Lecture I

Metric/connection compatibility

$E \longrightarrow (M, J), D, H$

Say D is unitary with respect to H if...

- For any $S, S' \in \Omega^0(E)$,

$$dH(S, S') = H(DS, S') + H(S, DS')$$

- In any local unitary frame the connection 1-form A is $\mathfrak{u}(n)$-valued, i.e.

$$A + \overline{A}^T = 0$$

Metric/connection compatibility

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Chern connection

Given $\overline{\partial}_E$ and H there is a unique connection, $D_{\overline{\partial}_E, H}$ such that

1. $D^{(0,1)}_{\overline{\partial}_E, H} = \overline{\partial}_E$, and
2. $D_{\overline{\partial}_E, H}$ is unitary with respect to H

Metric/connection compatibility

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Note: There is no reason why $D_{\overline{\partial}_E,H}$ should in general be flat

For any pair of connections D_1 and D_2

$$\begin{cases} D_1(fS) = (df)S + fD_1(S) \\ D_2(fS) = (df)S + fD_2(S) \end{cases}$$

the difference $\Phi = D_1 - D_2$ gives $\phi : \Omega^0(E) \rightarrow \Omega^1(E)$ with

$$\Phi(fS) = f\Phi(S)$$

i.e. $\boxed{\Phi \in \Omega^1(End(E))}$.

Modifications to $D_{\bar{\partial}_E, H}$

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Goal

Find $\Phi \in \Omega^1(End(E))$ so that $D = D_{\bar{\partial}_E, H} + \Phi$ is **flat**.

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Higgs bundles show how.....

Higgs (vector) bundles on Riemann surfaces

A **Higgs bundle** on a Riemann surface $\Sigma = (S, J)$ is a pair $(\mathcal{E}, \varphi)$ where

- $\mathcal{E} = (E, \overline{\partial}_E)$ is a rank n holomorphic bundle,
- $\varphi \in \Omega^{(1,0)}(End(\mathcal{E}))$ is holomorphic (i.e. $\overline{\partial}_E \varphi = 0$)

- impose $\varphi \wedge \varphi = 0$ if base has $\dim_{\mathbb{C}} > 1$
- impose $Tr(\varphi) = 0$ if $G = \mathrm{SL}(n, \mathbb{C})$

The Higgs field: $\varphi \in \Omega^{(1,0)}(End(\mathcal{E}))$ with $\overline{\partial}_E\varphi = 0$

- In a local holomorphic frame for $\mathcal{E}$, with local coordinate z on Σ,

$$\varphi = \phi(z)dz \ ,$$

where $\phi(z) \in \mathfrak{gl}(n, \mathbb{C})$ and $\frac{\partial}{\partial \bar{z}}\phi = 0$.

- Write $\varphi \in H^0(End(\mathcal{E}) \otimes K_\Sigma)$ where $K_\Sigma = (T^*\Sigma)^{1,0}$, or

$$\varphi : \mathcal{E} \to \mathcal{E} \otimes K_\Sigma$$

Example

If $\mathcal{E} = \Sigma \times \mathbb{C} = \mathcal{O}_\Sigma$ and φ is any holomorphic abelian differential, then $(\mathcal{E}, \varphi)$ is a rank 1 Higgs bundle.

[Teaser: Later we will see (rank 2) Higgs bundles with Higgs fields defined by quadratic differentials]

Recipe for a flat connection $(\mathcal{E} = (E, \overline{\partial}_E), \varphi \in H^0(End(\mathcal{E}) \otimes K_\Sigma))$

1. Take a metric H on E
2. Construct $D_{\overline{\partial}_E, H}$
3. Construct $\varphi^{*_H} \in \Omega^{(0,1)}(End(E))$ using $H(\varphi(u), v) = H(u, \varphi^{*_H}(v))$

   [If $\varphi = \phi(z)dz$ then $\varphi^{*_H} = \overline{\phi}^T(z)d\overline{z}$]

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$$\boxed{\nabla_H = D_{\overline{\partial}_E, H} + \underbrace{(\varphi + \varphi^{*_H})}_{\Omega^1(End(E))}}$$

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$$\nabla_H = D_{\overline{\partial}_E, H} + \underbrace{(\varphi + \varphi^{*_H})}_{\Omega^1(End(E))}$$

4. **[Challenge] pick H so that ∇_H is flat.**

- Denote curvature by $F_{\nabla_H} = \nabla_H^2$ and $F_{\overline{\partial}_E,H} = D^2_{\overline{\partial}_E,H}$

$$\left.\begin{array}{r} F_{\nabla_H} = 0 \\ \overline{\partial}_E\varphi = 0 \end{array}\right\} \Longleftrightarrow \boxed{\begin{array}{c} F_{\overline{\partial}_E,H} + [\varphi,\varphi^{*H}] = 0 \\ \overline{\partial}_E\varphi = 0 \end{array}} \quad \text{Hitchin's equation}$$

The immediate questions

1. For which $\rho : \pi_1(S) \to \mathrm{GL}(n,\mathbb{C})$ can we construct the corresponding flat bundle in this way?
2. What does existence of solutions say about the Higgs bundle, i.e. can we answer (1) purely in Higgs bundle terms?

Connections in the presence of metric $\boxed{D, H \text{ on } E \to S}$

A metric H on E

- reduces the structure group of E to $\mathrm{U}(n)$
- decomposes

$$End(E) = E_H(\mathfrak{u}(n)) \oplus E_H(\mathfrak{m})$$

$$\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{m} \quad (\mathfrak{m} = \{\text{Hermitian matrices}\})$$

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Then any connection D decomposes as $D = D_H \oplus \Phi$ with

- D_H unitary
- $\Phi \in \Omega^1(E_H(\mathfrak{m}))$

$$\boxed{F_D = 0} \Longleftrightarrow \boxed{\begin{array}{c} F_H + [\Phi, \Phi] = 0 \\ D_H(\Phi) = 0 \end{array}}$$

$$[F_D = D^2; \quad F_H = D_H^2]$$

Comparison with Hitchin equations

D, H on $E \longrightarrow \Sigma$

If $\Sigma = (S, J)$ then on (E, H):

$$\begin{aligned}\nabla_H &= (\overline{\partial}_E + D^{(1,0)}_{\overline{\partial}_E,H}) + (\varphi + \varphi^{*_H}) \\ &= D_H + \Phi\end{aligned} \quad \text{with} \quad \boxed{\begin{aligned}\Phi &= \varphi + \varphi^{*_H} \\ D_H &= D_{\overline{\partial}_E,H}\end{aligned}}$$

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$$\boxed{\begin{array}{c}F_{\nabla_H} = 0 \\ \text{on} \\ (\mathcal{E}, \varphi)\end{array}} \iff \boxed{\begin{array}{c}F_{\overline{\partial}_E, H} + [\varphi, \varphi^{*_H}] = 0 \\ \overline{\partial}_E \varphi = 0\end{array}}$$

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$$\boxed{\begin{matrix} F_{\nabla_H} = 0 \\ \text{on} \\ (\mathcal{E}, \varphi) \end{matrix}} \iff \boxed{\begin{matrix} F_{\overline{\partial}_E, H} + [\varphi, \varphi^{*H}] = 0 \\ \overline{\partial}_E \varphi = 0 \end{matrix}} \iff \boxed{\begin{matrix} F_H + [\Phi, \Phi] = 0 \\ D_H(\Phi) = 0 \\ D_H^*(\Phi) = 0 \end{matrix}} \iff \boxed{\begin{matrix} F_D = 0 \\ D_H^*(\Phi) = 0 \end{matrix}}$$

Using H and a J-compatible Riemannian metric to define D_H^* via

$$\Omega^0(E) \underset{D_H^*}{\overset{D_H}{\rightleftarrows}} \Omega^1(E)$$

Metrics in the presence of flat connections: D, H on $E \to S$

- $E = \left(\coprod_{\alpha \in I} U_\alpha \times \mathbb{C}^n\right) / \{g_{\alpha\beta} \in \mathrm{GL}(n, \mathbb{C})\}$
- $H \rightsquigarrow \begin{Bmatrix} \text{change of basis} \\ \text{on each fiber} \end{Bmatrix} \rightsquigarrow g_{\alpha\beta} \in \mathrm{U}(n)$

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- H defined by $\{h_\alpha : U_\alpha \to \mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)\}$ i.e.

$$\boxed{H \rightsquigarrow h \in \Omega^0(E(\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n))}$$

- D Flat $\rightsquigarrow \rho : \pi_1(S) \to \mathrm{GL}(n,\mathbb{C}) \rightsquigarrow E = \tilde{S} \times_\rho \mathbb{C}^n$
- $E(\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)) = \tilde{S} \times_\rho \mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$

$$
\begin{array}{ccc}
\tilde{S} \times (\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)) & \longrightarrow & \tilde{S} \times_\rho (\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)) \\
\boxed{\tilde{h}} \uparrow & & \uparrow h \\
\tilde{S} & \longrightarrow & S
\end{array}
$$

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$$\begin{array}{ccc} \tilde{S} \times (\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)) & \longrightarrow & \tilde{S} \times_\rho (\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)) \\ \boxed{\tilde{h}} \uparrow & & \uparrow h \\ \tilde{S} & \longrightarrow & S \end{array}$$

$$\boxed{\tilde{h} : \tilde{S} \to X = (\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n))}$$

The meaning of $D_H^*(\Phi) = 0$ $\boxed{D, H \text{ on } E \to S}$

- $H \rightsquigarrow \boxed{D = D_H + \Phi}$
- $F_D = 0 \rightsquigarrow \boxed{\tilde{h} : \tilde{S} \to X = (\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n))}$

- $\tilde{S}$ inherits metric from S
- $X = (\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n))$ has (invariant) metric
- $\tilde{h} : \tilde{S} \to X$ is $\pi_1(S)$-equivariant
- $\tilde{h} : \tilde{S} \to X$ is harmonic if it minimizes $\int_\Sigma |dh| dvol$

$$D_H^* \Phi = 0 \Leftrightarrow \tilde{h} : \tilde{S} \to X = (\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)) \text{ is harmonic}$$

[Corlette]

Theorem (Corlette)

Given a flat connection D on E, T.F.A.E

1. *E admits a harmonic metric*
2. *E admits a metric such that $D_H^*\Phi = 0$ (where $D = D_H + \Phi$)*
3. *D is reducible*
4. *The corresponding $\rho : \pi_1(S) \to \mathrm{GL}(n, \mathbb{C})$ is reductive.*

This answers our first question (for which representations can we construct the flat bundle using Higgs bundles); what does it mean?

Reductive representations

- $\rho : \pi_1(S) \to \mathrm{GL}(n, \mathbb{C})$ reductive $\Leftrightarrow \overline{\rho(\pi_1(S))} \subset \mathrm{GL}(n, \mathbb{C})$ is reductive
- $G \subset \mathrm{GL}(n, \mathbb{C})$ is reductive $\Leftrightarrow \mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$ is reductive
- $\mathfrak{g}$ is reductive $\Leftrightarrow$ the adjoint representation is completely reducible.

Example ($\mathrm{GL}(2, \mathbb{C})$)

If $\rho([\gamma]) = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ then ρ is not reductive.

Why we like reductivity

- $Hom(\pi_1(S), G) \subset \underbrace{G \times \cdots \times G}_{2g}$ is an algebraic subvariety
- $Rep(\pi_1(S), G)) = Hom(\pi_1(S), G)/G$ has bad orbits at non-reductive ρ!
- $Rep^{red}(\pi_1(S), G))$ is good.

Implications for $(\mathcal{E}, \varphi)$

$\boxed{\Sigma = (S, J)}$

$\rho : \pi_1(S) \to \mathrm{GL}(n, \mathbb{C})$ reductive

$\Leftrightarrow$ (E, D) admits harmonic metric

$\Leftrightarrow$ (E, D) admits metric such that $D_H^* \Phi = 0$

$\Leftrightarrow$ $(\mathcal{E}, \varphi)$ is a Higgs bundle which admits a metric satisfying Hitchin's equation

$\Leftrightarrow$ $\boxed{?}$

The answer involves stability, a property required to construct good moduli spaces of Higgs bundles. But first......

The degree of a complex line bundle on Σ $\boxed{L \longrightarrow \Sigma}$

- pick any point $p \in \Sigma$ and open disk $\mathcal{D} \subset \Sigma$ containing p

$$\boxed{L = \mathcal{D} \times \mathbb{C} \coprod (\Sigma - \{p\}) \times \mathbb{C} / \{g\}}$$

where

$$\begin{array}{ccc} \mathcal{D} \cap (\Sigma - \{p\}) & \xrightarrow{\ g\ } & \mathbb{C}^* \\ {\scriptstyle def.\ ret.} \downarrow & & \downarrow {\scriptstyle def.\ ret.} \\ S^1 & \xrightarrow{\ \hat{g}\ } & S^1 \end{array}$$

$\deg(L) = \deg(\hat{g})$ (winding number)

The degree of a complex vector bundle on Σ

$\boxed{E \to \Sigma}$

- Step 1: define $\det(E)$

E	$\det(E)$
$\{g_{\alpha\beta}\}$	$\{\det(g_{\alpha\beta})\}$

$$\deg(E) = \deg(\det(E))$$

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$$\deg(E) = \deg(\det(E))$$

- Chern-Weil:

$$\deg(E) = \frac{\sqrt{-1}}{2\pi} \int_\Sigma Tr(F_D) = \int_\Sigma c_1(E)$$

where D is any connection on E. In particular....

The degree of a complex vector bundle on Σ $\boxed{E \to \Sigma}$

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where D is any connection on E. In particular....

- ...if $F_D = 0$ then $\deg(E) = 0$

Stability of Higgs bundles on Σ $(\mathcal{E}, \varphi)$

- Look at holomorphic subbundles $\mathcal{E}' \subset \mathcal{E}$ preserved by φ, i.e. $\varphi(\mathcal{E}') \subset \mathcal{E}' \otimes K_\Sigma$.

Definition

- A rank n Higgs bundle on Σ, $(\mathcal{E}, \varphi)$ is **(semi)stable** if

$$\frac{\deg(\mathcal{E}')}{\mathrm{rank}(\mathcal{E}')} (\leq) < \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}$$

  for all φ-invariant subbundles $\mathcal{E}' \subset \mathcal{E}$.
- A semistable Higgs bundle is **polystable** if it decomposes as a direct sum of stable Higgs bundles.

- Remove 'φ-invariant' to get definitions for holomorphic bundles
- Comes from GIT condition for a good moduli space of objects

back to the meaning of the Hitchin equations...

The Hitchin-Kobayashi correspondence

$(\mathcal{E}, \varphi)$

Theorem (Hitchin, Simpson)

Let $(\mathcal{E}, \varphi)$ be a rank n, degree zero Higgs bundle on Σ. Then $\mathcal{E}$ admits a metric satisfying Hitchin's equation if and only if $(\mathcal{E}, \varphi)$ is polystable.

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Let $(\mathcal{E}, \varphi)$ be a rank n, degree zero Higgs bundle on Σ. Then $\mathcal{E}$ admits a metric satisfying Hitchin's equation if and only if $(\mathcal{E}, \varphi)$ is polystable.

- We can remove the degree zero assumption if we adjust the equation:

$$F_{\bar{\partial}_E, H} + [\varphi, \varphi^{*H}] = -2\pi\sqrt{-1}\left(\frac{\deg E}{\operatorname{rank} E}\right)\omega$$

- The result generalizes the Hitchin-Kobayashi correspondence for bundles [Narasimhan-Seshadri, Lubke, Uhlenbeck-Yau, Donaldson..]
- The set of isomorphism classes of polystable objects defines $\mathcal{M}_{Higgs}(\Sigma, \mathrm{GL}(n, \mathbb{C}))$, the moduli space of polystable degree zero Higgs bundles on Σ

[This answers our second question.]

License to use Higgs bundles

$$\mathrm{Rep}^{red}(\pi_1(S), \mathrm{GL}(n, \mathbb{C})) \leftrightarrow \mathcal{M}_{Higgs}(\Sigma, \mathrm{GL}(n, \mathbb{C}))$$

End of Lecture II