Introduction to Higgs bundles

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Disclaimer

These slides are precisely as they were during the lectures on July 23, 25, 27, 2012. As such, they contain several omissions and inaccuracies, in both the mathematics and the attributions. Some of these, it must be admitted, are blemishes which reflect the author's limitations, but others reflect the fact that:

- The slides formed but one part of the lectures. They were accompanied by verbal commentary designed to explain and embellish the contents of the slides
- This is not a paper. Any talk has to strike a balance between accuracy and accessibility. This balance inevitably involves the inclusion of some half-truths and/or white lies.

The author apologizes to anyone who is in any way led astray by the inaccuracies or slighted by the omissions.

Goals and plan for this mini-course

- What are Higgs bundles?
- How do they relate to surface group representations?
- What do we gain by taking the Higgs bundle point of view?

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The Plan:

1. (Lectures I and II)Description of surface group representations from a bundle perspective, with necessary background to define Higgs bundles and to see their relation to the representations
2. (Lecture III) Examples and properties of Higgs bundles

The main dramatis personae

- S a closed surface of genus g
- G a Lie group (mostly $\mathrm{GL}(n, \mathbb{C})$ for us)

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Representations $\rho : \pi_1(S) \longrightarrow G$

- $\pi_1(S) =< a_1, \ldots, a_g, b_1, \ldots, b_g \mid \prod_i (a_i b_i a_i^{-1} b_i^{-1}) = 1 \ >$
- $\rho : \begin{cases} a_i \mapsto \alpha_i \\ b_i \mapsto \beta_i \end{cases}$ such that $\prod_i (\alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}) = 1$

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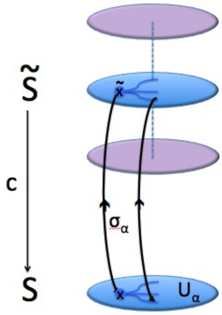
Higgs bundles on $\Sigma = (S, J)$, i.e. pairs $(\mathcal{E}, \varphi)$

- $\mathcal{E} \longrightarrow \Sigma$ a rank n holomorphic bundle
- $\varphi : \mathcal{E} \longrightarrow \mathcal{E} \otimes (T^{1,0}\Sigma)^*$, i.e. $\varphi \in H^0(End(\mathcal{E} \otimes K_\Sigma)$

From $\boxed{\rho : \pi_1(s) \longrightarrow G}$ to $\boxed{(\mathcal{E}, \varphi)}$ (with $G = \mathrm{GL}(n, \mathbb{C})$)

$$
\begin{tikzcd}
\boxed{\rho : \pi_1(S) \to G} \arrow[r, leftrightarrow] & \boxed{\text{local system on } S} \arrow[d, leftrightarrow] \\
\boxed{(\mathcal{E}, \varphi) \text{ on } \Sigma} & \boxed{\text{bundle with flat connection on } S} \arrow[l]
\end{tikzcd}
$$



- Take the universal cover $\tilde{S} \xrightarrow{c} S$
- path lifting defines local sections
- $\pi_1(S)$ acts on $\tilde{S}$ preserving fibers of c

$$\boxed{\begin{array}{c} \tilde{S} \times_\rho G \\ \big\downarrow \pi \\ S \end{array}}$$

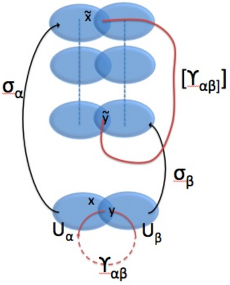
Use $\rho : \pi_1(S) \to G$ to construct $\tilde{S} \times G/\pi_1(S) = \tilde{S} \times_\rho G$[A local system]

Over U_α

$$\tilde{S} \times_\rho G|_{U_\alpha} \simeq U_\alpha \times G$$
$$[\sigma_\alpha(x), g] \leftrightarrow (x, g)$$

Over $U_\alpha \cap U_\beta$

$$\begin{tikzcd}
{[\sigma_\beta(x), g]} \arrow[r] \arrow[d, "\sigma_\alpha(x) = {[\gamma_{\alpha\beta}]}\sigma_\beta(x)"'] & (x, g) \arrow[d, "\rho{[\gamma_{\alpha\beta}]}"] \\
{[\sigma_\alpha(x), \rho[\gamma_{\alpha\beta}]g]} & (x, \rho[\gamma_{\alpha\beta}]g) \arrow[l]
\end{tikzcd}$$



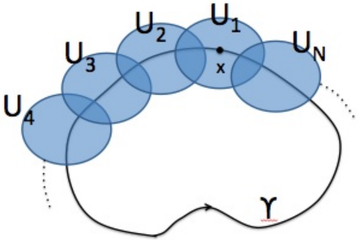
$\tilde{S} \times_\rho G$ is a G-Local System described by:

- $\{U_\alpha\}_{\alpha \in I}$ (open cover of S)
- $\{g_{\alpha\beta} = \rho([\gamma_{\alpha\beta}])\}$ (transition data satisfying $\{g_{\alpha\beta} g_{\beta\delta} g_{\delta\alpha} = 1\}$)

Monodromy

Any G-Local System defines a representation $\rho : \pi_1(S) \to G$ by monodromy:

- cover loop γ by $U_1, U_2, \ldots, U_k$
- define $\rho([\gamma]) =$ $g_{N(N-1)}g_{(N-1)(N-2)} \cdots g_{21}$



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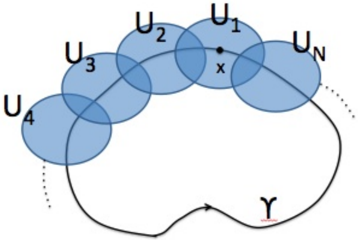

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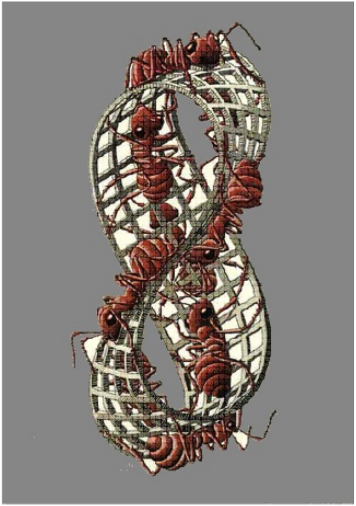
Coming up:

- G-Local System = bundle with flat connection
- monodromy = holonomy

Vector Bundles



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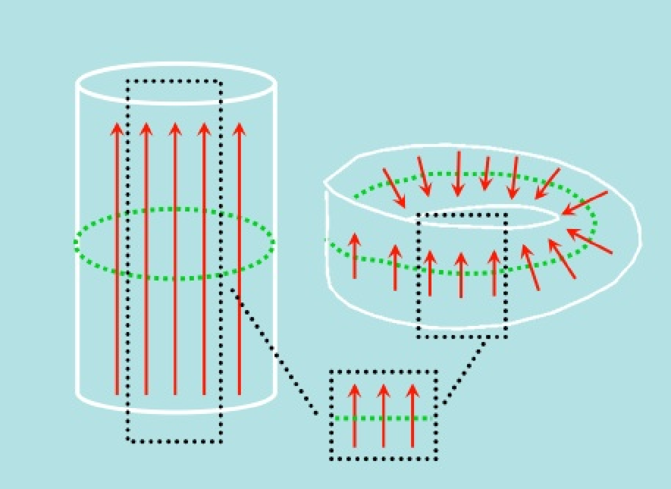

Bundle Basics: I. Vector bundles over M

$$\begin{tikzcd} V \arrow[r] & E \arrow[d] \\ & M \end{tikzcd}$$

- a cover $\{U_\alpha\}_{\alpha\in I}$ for M
- local trivializations $E|_{U_\alpha} \simeq U_\alpha \times V$
- transition functions (gluing data): $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$

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$$E = \left(\coprod_{\alpha \in I} U_\alpha \times V\right)/\sim \quad \text{where } (x, v_\alpha) \sim (x, g_{\alpha\beta}(x) v_\beta)$$

- Cocycle condition on triple overlaps:

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$$

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Example

If $g_{\alpha\beta} = I$ for all $U_\alpha \cap U_\beta \neq \emptyset$ then $E = M \times V$

Bundle Basics: Principal and associated bundles

- $\{U_\alpha\} + \{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\} = E_G$

Principal G-bundle

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- $E_G + \{r : G \to \mathrm{GL}(V)\} = E_G(V)(\text{or } E_V)$

Associated V-bundle

$$E_G(V) = (\coprod_{\alpha \in I} U_\alpha \times V)/\sim \;\; \text{where } (x, v_\alpha) \sim (x, r(g_{\alpha\beta}(x)) v_\beta)$$

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Example ($G = \mathrm{GL}(n, \mathbb{C})$)

$V = \mathfrak{gl}(n, \mathbb{C})$; $r = \text{adjoint} \implies E_G(V) = End(E)$

G-Local Systems as Principal Bundles

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For a bundle E, a choice of local trivializations for which $dg_{\alpha\beta} = 0$ is called a **flat structure** on the bundle. A bundle together with a flat structure is called a **flat bundle**

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- (With $M = S$)

$\boxed{\text{Representations } \pi_1(S) \to G}$ correspond to $\boxed{\text{flat principal } G\text{-bundles}}$

The next step....

$$\begin{tikzcd}
\boxed{\rho : \pi_1(S) \to G} \arrow[r, leftrightarrow] & \boxed{\begin{array}{c}\text{local system} = \text{flat bundle} \\ \text{on } S\end{array}} \arrow[d, leftrightarrow] \\
\boxed{\begin{array}{c}(\mathcal{E}, \varphi) \\ \text{on } \Sigma\end{array}} & \boxed{\begin{array}{c}\text{bundle with} \\ \text{flat connection} \\ \text{on } S\end{array}} \arrow[l]
\end{tikzcd}$$

..provide the solution to the following:

1. At a point $q \in E$ which directions are "horizontal" or "parallel to the base"

Connections on vector bundles...

$E \xrightarrow{\pi} M$

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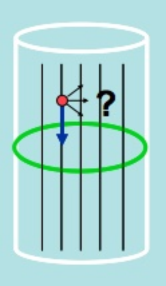
..provide the solution to the following:

1. At a point $q \in E$ which directions are "horizontal" or "parallel to the base"
2. How to compare fibers over different points in the base?
3. How to measure variations in fiber direction with respect to motion along base?

Horizontal directions

$$E \xrightarrow{\pi} M$$



$$\begin{tikzcd} 0 \arrow[r] & V_q \arrow[r] & T_qE \arrow[r, "\pi_*"] & T_{\pi(q)}M \arrow[r] \arrow[l, bend left, "?"] & 0 \end{tikzcd} \quad ?$$

- Vertical directions lie in $V_q = Ker\pi_*$.
- How do we identify a complementary subspace 'parallel' to $T_{\pi(q)}M$?

Variations of sections

Local picture - using trivialization Ψ_α over $U_\alpha \subset M$...

$$\begin{tikzcd}
E|_{U_\alpha} \arrow[r, "\Psi_\alpha"] \arrow[d] & U_\alpha \times \mathbb{C}^n \arrow[d] & (x, \vec{s}_\alpha(x)) \\
U_\alpha \arrow[u, bend left, "s"] & U_\alpha \arrow[u, bend right, "s_\alpha"'] & x \arrow[u]
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..or, in terms of local frame $\{\mathbf{e}^i_\alpha(x) = \Psi^{-1}_\alpha(x, e^i)\}$,

$$S(x) = \sum_{i=1}^{n} s^i_\alpha(x)\mathbf{e}^i_\alpha(x)$$

- $ds^i_\alpha(x)$ measures variation of coefficients (i.e. local sections)
- *How do we take into account variation of the local frame ?*

What a connection is:

1. In terms of local frames $\{\mathbf{e}^i_\alpha(x)\}$: $\mathfrak{gl}(\mathfrak{n},\mathbb{C})$-valued 1-forms related by

$$A_\alpha = g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} + g_{\alpha\beta} d g_{\alpha\beta}^{-1}$$

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2. Globally: a $\mathbb{C}$-linear operator satisfying a Leibniz rule

$$D : \Omega^0(E) \to \Omega^1(E)$$
$$D(fS) = (df)S + fDS$$

[$\Omega^k(E)$= k-forms with values in E, $f \in C^\infty(M), S \in \Omega^0(E)$]

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$$\boxed{D\mathbf{e}^i_\alpha(x) = [A_\alpha(x)]_{ji}\mathbf{e}^j_\alpha(x)}$$

Definition

With respect to connection D a section $S \in \Omega^0(E)$ is called

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- $DS = 0$ is an overconstrained system of PDE's
- Given $S(0) = S(\gamma(0))$, (**??**) has a unique solution along any curve.

Parallel sections along curves define **horizontal lifts** to E of curves in M

To split

$$
\begin{tikzcd}
0 \arrow[r] & V_q \arrow[r] & T_q E \arrow[r, "\pi_*"] & T_{\pi(q)} M \arrow[r] \arrow[l, bend left, "?"] & 0
\end{tikzcd}
$$

use horizontal lifts:

- For $v \in T_x M$ pick a path γ such that $v = \dot{\gamma}(0)$,
- Define

$$\boxed{v \mapsto \dot{\gamma}_q^h(0)}$$

where $\gamma_q^h(t)$ is the horizontal lift of γ through q to get

$H_q = \{$ tangents to horizontal lifts of curves through $x = \pi(q)\}$

- $T_q E = V_q \oplus H_q$

To compare E_{x_1} and E_{x_2}

- Pick a path γ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$
- For each $q \in E_{x_1}$ take

$$\boxed{q \mapsto \gamma_q^h(1)}$$

where $\gamma_q^h(t)$ is the horizontal lift of γ through q to get

Get a linear map

$$P_\gamma : E_{x_1} \to E_{x_2}$$

called **Parallel Transport along** γ.

Holonomy around loops in M

$\boxed{E \xrightarrow{\pi} M\,,\, D}$

- $\gamma : [0,1] \to M$ with $\gamma(0) = x = \gamma(1)$

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Definition

Parallel transport $P_\gamma : E_x \to E_x$ defines a linear map on E_x called the **holonomy around** γ

- In general the holonomy map depends on the loop γ
- Under special conditions the map depends only on the homotopy class $[\gamma] \in \pi_1(S, x)$....

Curvature of a connection

$\boxed{E \xrightarrow{\pi} M \ , D}$

What it tells us....

- $H_q \subset T_q M$ defines the **horizontal distribution** $\mathfrak{D} \subset TM$.

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- $H_q \subset T_qM$ defines the **horizontal distribution** $\mathfrak{D} \subset TM$.
  When is this integrable?
- Local sections $\{S_1, S_2, \ldots S_n\}$ such that
  1. $DS_i = 0$
  2. $\{S_1(x), S_2(x), \ldots S_n(x)\}$ linearly independent at all x

  define a **horizontal local frame**.
  When can we find **horizontal local frames**?

Curvature of a connection

What it is....

- In a local frame: $(D = d + A)$

$$F_D = dA + A \wedge A$$

  A matrix-valued 2-form

- As a global operator: ($\Omega^0(E) \xrightarrow{D} \Omega^1(E) \xrightarrow{D} \Omega^2(E)$)

$$F_D = D \cdot D = D^2$$

  A section in $\Omega^2(End(E))$

Flat connections

$E \xrightarrow{\pi} M$, D

Definition

A connection D on E is **flat** if $F_D = 0$

Example

If E has local trivializations with $dg_{\alpha\beta} = 0$, i.e. a flat structure, then

$$A_\alpha = 0$$

defines a connection. The curvature clearly vanishes.

Implications of flatness

- Horizontal local frames $\implies$ locally constant transition functions, i.e.

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$\boxed{E \xrightarrow{\pi} M \,,\, F_D = 0}$

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Flat connections define flat structures

- Holonomy around a loop γ depends only on $[\gamma] \in \pi_1(M)$, i.e.

Holonomy defines $\rho : \pi_1(M) \to \mathrm{GL}(n, \mathbb{C})$

Summary..so far

$$
\begin{tikzcd}
\boxed{\rho : \pi_1(M) \to \mathrm{GL}(n,\mathbb{C})} \arrow[d, shift right] & \boxed{\text{Holonomy representation}} \arrow[l] \\
\boxed{\text{Flat } \mathrm{GL}(n,\mathbb{C})\text{-bundles}} \arrow[u, shift right, "\text{monodromy}"'] \arrow[r, leftrightarrow] & \boxed{E \to M \text{ with } F_D = 0} \arrow[u]
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Summary..so far

$$\begin{array}{ccc}
\boxed{\rho : \pi_1(M) \to \mathrm{GL}(n, \mathbb{C})} & \longleftarrow & \boxed{\text{Holonomy representation}} \\
\downarrow\uparrow \text{ monodromy} & & \uparrow \\
\boxed{\text{Flat } \mathrm{GL}(n, \mathbb{C})\text{-bundles}} & \longleftrightarrow & \boxed{E \to M \text{ with } F_D = 0}
\end{array}$$

next: how to build flat connections....using

- complex structures
- metric structures

Complex structures on a manifold M

- **complex** coordinate charts $\Psi_\alpha : U_\alpha \to \mathbb{C}^m$
- **holomorphic** coordinate transformations: $\Psi_\beta \Psi_\alpha^{-1} : \mathbb{C}^m \to \mathbb{C}^m$
- denote by J

Example ($M = S$)

- $\dim_{\mathbb{C}} = 1$
- $(S, J) = \Sigma$, a Riemann surface
- equivalent to choice of conformal structure

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With respect to J:

$$\begin{array}{ccccc} \Omega^1(M,\mathbb{C}) & = & \Omega^{(0,1)}(M,\mathbb{C}) & \oplus & \Omega^{(1,0)}(M,\mathbb{C}) \\ \{dx_1, \ldots dx_{2m}\} & \to & \{dz_1, \ldots, dz_m\} & + & \{d\overline{z}_1, \ldots d\overline{z}_m\} \end{array}$$

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Holomorphic structures on a vector bundle $E \longrightarrow M$...

...can be described in three ways:

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- a 'partial connection', i.e. $\overline{\partial}_E : \Omega^0(E) \to \Omega^{(0,1)}(E)$ such that

$$\overline{\partial}_E(fS) = (\overline{\partial} f)S + f\overline{\partial}_E S \ (\mathit{Leibniz})$$
$$\overline{\partial}_E^2 = 0 \ (\mathit{Integrability})$$

$\mathcal{E} = (E, \overline{\partial}_E)$ defines a holomorphic bundle

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$$\overline{\partial}_E^2 = 0 \ (\mathit{Integrability})$$

$\mathcal{E} = (E, \overline{\partial}_E)$ defines a holomorphic bundle

A section $S \in \Omega^0(\mathcal{E})$ is holomorphic iff

- $\overline{\partial}_E S = 0$
- In local frames $\{\mathbf{e}^i_\alpha\}$ with $\overline{\partial} g_{\alpha\beta} = 0$, if $S(z) = \sum_{i=1}^n s^i_\alpha(z)\mathbf{e}^i_\alpha(z)$ then $\overline{\partial} s^i_\alpha(z) = 0$.

Holomorphic versus flat structures

$E \to (M, J)$

	Flat	Holomorphic
Transition functions	$dg_{\alpha\beta} = 0$	$\overline{\partial} g_{\alpha\beta} = 0$
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$E \to (M, J)$

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- Flat $\Longrightarrow$ holomorphic
- On a Riemann surface, $\bar\partial_E^2 = 0$ is automatic $(d\bar z \wedge d\bar z = 0)$.
- $\bar\partial_E$ defines the $(0,1)$ part of a connection. Can complete to a connection using a **hermitian metric**......

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- Defines a reduction of structure group from $\mathrm{GL}(n, \mathbb{C})$ to $\mathrm{U}(n)$.

End of Lecture I