INTRODUCTION TO HIGGS BUNDLES ON RIEMANN SURFACES

Problems

1.

Let $\tilde{S}$ be the universal cover for S and let $p : \tilde{S} \to S$ be the covering map. Given a group homomorphism $\rho : \pi_1(S) \to G$, show that $\pi_1(S)$ acts on $\tilde{S} \times G$ and that the quotient by this action (denoted the $\tilde{S} \times_\rho G$) can be viewed as the total space of a G-bundle over S. [Hint: Consider the map $[\tilde{x}, g] \mapsto (p(x), g)$.]

Show that the bundle has local trivializations over an atlas of open sets $\{U_\alpha \subset S\}$for which the transition functions on non-empty intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$ are *locally constant* maps $g_{\alpha\beta} : U_{\alpha\beta} \to G$. That is, show that $\tilde{S} \times_\rho G$ defines a flat bundle (or local system).

2.

Let E_1 and E_2 be vector bundles over a closed surfaces S. Suppose that with respect to an open cover $\{U_\alpha\}$ for S the bundles are described by transition functions $\{g^{(1)}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n_1, \mathbb{C})\}$ and $\{g^{(2)}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n_2, \mathbb{C})\}$.

(1) Let $E_1 \oplus E_2$ be the vector bundle of rank $n_1 + n_2$ described by transition functions

$$g_{\alpha\beta} = \begin{bmatrix} g^{(1)}_{\alpha\beta} & 0 \\ 0 & g^{(2)}_{\alpha\beta} \end{bmatrix}$$

Prove that $\deg(E_1 \oplus E_2) = \deg(E_1) + \deg(E_2)$.

(2) Let L be a line bundle on S, described by transition functions $\{l_{\alpha\beta}\}$. Let L^{-1} be the line bundle described by transition functions $\{l^{-1}_{\alpha\beta}\}$.
Prove that $\deg(L^{-1}) = -\deg(L)$

(3) Let $E_1 \otimes L$ be the rank n_1 vector bundle described by transition functions $\{g^{(1)}_{\alpha\beta} l_{\alpha\beta}\}$. Compute $\deg(E_1 \otimes L)$ in terms of $\deg(E_1)$ and $\deg(L)$.

(4) Generalize (3) to the case $E_1 \otimes E_2$, i.e. to the tensor product of two vector bundles.

3.

For each usual operation on vector spaces (dual, direct sum, tensor product, homomorphism, etc.) define the corresponding operation on vector bundles. Explicitly give the transition functions.

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4.

For a complex vector space V , what is the conjugate vector space V ? Give the corresponding operation for a vector bundle E.

5.

Prove that every bundle on the interval [0,1] is trivial (i.e. isomorphic to the trivial bundle). Prove that every $\mathbb{C}^n$-bundle over S^1 is trivial.

6.

Show that every vector bundle admits smooth global sections.

7.

Prove that the structure group of a principal G-bundle E_G reduces to $K \subset G$ if and only if the associated fiber bundle $E_G \times_G (G/K)$ has a global section.

8.

Show that a principal G-bundle admits a global section if and only if the bundle is trivializable.

9.

Let $E_G \to X$ be a flat G-bundle. Let

$$\gamma_i : [0,1] \to X \ , i = 1,2$$

be homotopic paths in X with the same endpoints, i.e. such that

$$\gamma_1(0) = \gamma_2(0)$$
$$\gamma_1(1) = \gamma_2(1) \ .$$

Let

$$\tilde{\gamma}_i : [0,1] \to E_G \ , i = 1,2$$

be parallel lifts of γ_i to E_G, i.e. suppose that with respect to any flat local trivialization $\tilde{\gamma_i}(t)$ is given locally by $\tilde{\gamma_i}(t) = (\gamma_i(t), g_i)$ for some $g_i \in G$.

Prove that if $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0)$ then $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$ and hence that the monodromy around a loop in X depends only on the homotopy class of the loop.

10.

Let $V \to X$ be a vector bundle given by local trivializations

$$\psi_\alpha : U_\alpha \times \mathbb{R}^n \to E|_{U_\alpha}$$

and transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R}) \ .$$

Let $s : X \to V$ be the global section defined by local sections

$$s_\alpha : U_\alpha \to U_\alpha \times \mathbb{R}^n \ .$$

Show that in general the locally defined $\mathbb{R}^n$-valued 1-forms $\{ds_\alpha\}$ do not define a global section of $V \otimes T^*X$, but that they do so if the transition functions are locally constant.

11.

Let Σ be a closed orientable surface and let G be a connected Lie group. Show that an fiber bundle $E \to \Sigma$ with structure group G can be trivialized over $\Sigma - \{p\}$ where p is any point in Σ. Hence show that E can be described as a two-patch bundle with one transition function defined by a map $g : S^1 \to G$.

12.

Let L and M be holomorphic line bundles over a Riemann surface Σ. Suppose that with respect to a local trivializations over open sets in a cover $\{U_\alpha\}$ for Σ, the bundles are defined by transition functions $\{l_{\alpha\beta}\}$ and $\{m_{\alpha\beta}\}$ respectively.

- Give the transition functions which define the line bundles $L \otimes M$ (sometimes written simply as LM.
- Prove that a holomorphic map $f : L \to M$ defines a holomorphic section of $L^{-1}M$ and vice versa.
- Let $\mathcal{O}$ be the trivial line bundle on Σ, i.e. the bundle defined by transition functions $\{o_{\alpha\beta} = 1\}$. Prove that the locally defined functions $I_\alpha : U_\alpha \to \mathbb{C}$ where $I_\alpha(x) = 1$ define a section of $\mathcal{O}$. Prove that all holomorphic sections of $\mathcal{O}$ are constant multiples of the section I.

13.

Compute the degree of the tautological line bundle over $\mathbf{CP}^1$.

14.

Using holomorphic local frames, describe the connection 1-form for the Chern connection on a holomorphic bundle with a hermitian metric.

15.

Let E_G be a flat bundle. Fill in the details defining the monodromy action of $\pi_1(X)$ using locally consant ("parallel") sections and hence show that this defines a homomorphism from $\pi_1(X)$ to G.

16.

Let $\{U_\alpha\}$ be an atlas of co-ordinate charts for the surface Σ, with coordinates $\{x_\alpha, y_\alpha\}$ defined on U_α. Let $\{\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha}\}$ be the corresponding local frame for the (real) tangent bundle over U_α, i.e. $T\Sigma|_{U_\alpha}$ and let $J_\alpha : T\Sigma|_{U_\alpha} \to T\Sigma|_{U_\alpha}$ be the linear transformation given with respect to this frame by the matrix

$$J_\alpha = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

Show that the local transformations $\{J_\alpha\}$ define a global transformation $J : T\Sigma \to T\Sigma$ if and only if the coordinate transformations $(x_\alpha, y_\alpha) \mapsto (x_\beta, y_\beta)$ all satisfy the Cauchy-Riemann equations, i.e. if and only if

$$\frac{\partial x_\beta}{\partial x_\alpha} = \frac{\partial y_\beta}{\partial y_\alpha}$$
$$\frac{\partial x_\beta}{\partial y_\alpha} = -\frac{\partial y_\beta}{\partial x_\alpha}$$

wherever $U_\alpha \cap U_\beta \neq \emptyset$.

Let $T_{\mathbb{C}}\Sigma = T\Sigma \otimes \mathbb{C}$ be the complexified tangent bundle. If on U_α you make a complex gauge transformation to a new frame $\{\partial_{z_\alpha}, \partial_{\overline{z}_\alpha}\}$ where

$$\partial_{z_\alpha} = \frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}$$
$$\partial_{\overline{z}_\alpha} = \frac{\partial}{\partial x_\alpha} + i\frac{\partial}{\partial y_\alpha}$$

compute the transition function on $U_\alpha \cap U_\beta$ with respect to the new frames. Hence show that $T_{\mathbb{C}}\Sigma$ splits as a direct sum of complex line bundles, where $\{\partial_{z_\alpha}$ define local frames for one summand and $\{\partial_{\overline{z}_\alpha}$ define local frames for the other. Verify that these line subbundles are the $\pm\sqrt{-1}$ eigen-subbundles for J.

17.

Let $\mathcal{E}$ be a rank two, degree zero holomorphic bundle on the closed Riemann surface Σ. Show that either

(A) $\deg(L) \leqslant 0$ for all holomorphic line subbundles $L \subset \mathcal{E}$,

or

(B) there is a unique line subbundle $L \subset \mathcal{E}$ with $\deg(L) > 0$.

18.

Let L be a holomorphic line bundle on a Riemann surface Σ. Suppose that L admits a non-zero holomorphic section. Prove that $\deg(L) = 0$.

19.

If the Higgs bundle $(L \oplus L^{-1}, \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix})$ is isomorphic to $(L' \oplus L'^{-1}, \begin{bmatrix} 0 & \beta' \\ \gamma' & 0 \end{bmatrix})$ (with $\deg(L)$ and $\deg(L')$ both positive) then prove that $L = L'$, and $\beta = \lambda\beta'$, $\gamma = \lambda^{-1}\gamma'$ with $\lambda \in \mathbb{C}^*$