Exposition of *The ends of the moduli space of Higgs bundles*

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This note is an exposition of [MSWW1] and is organised as follows. First section is an introduction setting the stage and introducing the relevant structures and the main theorem of [MSWW1]. Secondly we present constructions of solutions to the anti-self-dual equations and its dimensional reductions to put [MSWW1] in a larger context. Thirdly we go into the nuts and bolts of the paper; this includes a description of a so called *limiting configuration* and desingularisation of these, followed by a perturbation argument to obtain exact solutions to the Higgs bundle equations. Lastly we describe a different more natural complex geometric construction of the limiting configurations which simplify parts of [MSWW1].

1 Introduction

First let us set some notation straight. We denote by X a smooth compact Riemann surface of genus $g \geq 2$ and E a Hermitian vector bundle of rank two. The *Higgs bundle equations* are equations for a pair (A, Φ) of a unitary connection on E and Φ an $\operatorname{End} E$-valued $(1,0)$-form on X. We will fix a background connection A_0 on E and only consider connections A which induce the same connections on $\det E$ as A_0, that is $A = A_0 + \alpha$ where $\alpha \in \Omega^1(\mathfrak{su}(E))$. We furthermore require that the Higgs field is trace-free, i.e. $\operatorname{Tr} \Phi = 0$. In this case the Higgs bundle equations are

$$F_A^{\perp} + [\Phi, \Phi^*] = 0 \quad \text{and} \quad \bar{\partial}_A \Phi = 0, \tag{1}$$

where Φ^* is computed with respect to the Hermitian metric on E. The first equation looks a bit different to what is often presented, namely

$$F_A + [\Phi, \Phi^*] = i\omega\mu$$

where ω is a Kähler form on X and μ is the slope of E, which in this case is $\frac{1}{2}\deg(E)$. If A is a unitary connection then its curvature decomposes as

$$F_A = F_A^{\perp} + \frac{1}{2}\operatorname{Tr}(F_A) \otimes \operatorname{Id}_E,$$

along with the Lie algebraic decomposition $\mathfrak{u}(E) = \mathfrak{su}(E) \oplus i\mathbb{R}$. Notice that $\frac{1}{2}\operatorname{Tr}(F_A)$ is precisely the curvature of the induced connection on $\det E$. If we require the induced connection on $\det E$ to be fixed we can remove this term from the equations and end up with (1). We say that $F_A^{\perp}$ is the trace-free part of A relative the background connection A_0.

We can form a moduli space of solutions to equations (1),

$$\mathcal{M}_{\mathrm{SL}}^{gauge}(d) = \{(A_0 + \alpha, \Phi) \,|\, \text{Solutions of (1)}\}/\Gamma(\mathrm{SU}(E)).$$

where $\mathrm{SU}(E)$ denotes the bundle of unitary endomorphisms of E with determinant one and $\Gamma(\mathrm{SU}(E))$ is the sections of this bundle, also called the special unitary gauge group. The number d referes to the degree of the bundle, which is fixed.

Since E is a Hermitian vector bundle a unitary connection A is equivalent to $\bar\partial$ a $\bar\partial$-operator compatible with the Hermitian metric on E and thus the pair (A, Φ) can be seen as a pair $(\bar\partial, \Phi)$. If the latter description is used there is an algebro-geometric definition of a moduli space where the Higgs bundle equations are replaced with a stability condition for a pair $(\bar\partial, \Phi)$ where Φ is $\bar\partial$-holomorphic

$$\mathcal{M}_{\mathrm{SL}}(d) = \{(\bar\partial, \Phi) \,|\, (\bar\partial, \Phi) \text{ is polystable, } \bar\partial \text{ induce } \bar\partial_{\det E}, \operatorname{Tr}(\Phi) = 0\}/\Gamma(\mathrm{SL}(E)).$$

Here $\mathrm{SL}(E)$ is the bundle of automorphisms of E with determinant one, and $\Gamma(\mathrm{SL}(E))$ is the sections, also called the special complex gauge group.

The key relation between the two moduli spaces is the following theorem first proved by Hitchin in rank two, [H2], and shortly after by Simpson in general, [S].

Theorem 1.1 (Hitchin/Simpson). *Let E be a fixed Hermitian complex vector bundle. In the $\Gamma(\mathrm{SL}(E))$ gauge orbit of the Higgs bundle $(\bar\partial, \Phi)$ there is a $\bar\partial$-operator whose associated Chern connection satisfies the Higgs bundle equations* (1) *if and only if $(\bar\partial, \Phi)$ is polystable.*

The theorem gives a diffeomorphism

$$\mathcal{M}_{\mathrm{SL}}(d) \simeq \mathcal{M}_{\mathrm{SL}}^{gauge}(d).$$

The stable locus of $\mathcal{M}_{SL}(d)$ is smooth, open, dense, and carries a hyperkähler structure which is complete when the degree is odd (in general it is when the rank and degree are coprime). Understanding the asymptotic behaviour of this metric towards infinity in the moduli space is the ultimate goal of [MSWW1].

The space of $\bar\partial$-operators is affine and modelled on $\mathcal{A} = \Omega^{0,1}(\operatorname{End} E)$. A Higgs field is an element of $\Omega^{1,0}(\operatorname{End} E)$ and an infinitesimal deformation is therefore an element of $\bar{\mathcal{A}} = \Omega^{1,0}(\operatorname{End} E)$. The Higgs bundle equations give a set of equations cutting out a subspace of $\mathcal{A} \times \bar{\mathcal{A}}$ defining the tangent space at a point $(\bar{\partial}, \Phi)$. The space $\mathcal{A} \times \bar{\mathcal{A}}$ itself carries a natural Kähler metric using the L^2-metric induced by the Hermitian metric on E,

$$g((\alpha, \varphi), (\beta, \psi)) = 2i \int_X \mathrm{Tr}(\alpha^* \wedge \beta + \varphi \wedge \psi^*).$$

The moduli space $\mathcal{M}_{\mathrm{SL}}(d)$ has been intensely studied by many since Hitchin's landmark paper [H2] and quite a lot is known about the moduli space, e.g. the topology. However, surprisingly little is known about the asymptotic behaviour of the hyperkähler metric towards infinity in the moduli space. The motivation for the paper [MSWW1] is to provide more information about the asymptotic behaviour of the metric. In the paper they give a constructive proof of the Hitchin–Simpson theorem when the norm of the Higgs field Φ is large and $\det \Phi$ has simple zeros.

An important ingredient in their proof is the notion of a *limiting configuration*, see Section 5. As the subset of the moduli space where Higgs fields are simple is open and dense, it follows from their proof of the Hitchin–Simpson theorem that the boundary of this open dense subset are the limiting configurations. The limiting configurations constitute a complex torus, and with an insight of Hitchin, this torus is related to a Prym variety, see Section 9.1.

The implications for the hyperkähler metric from the new proof of the Hitchin–Simpson theorem and the description of the "top boundary stratum" are indications that the metric is *semi-flat* on the dense open set of Higgs fields with simple zeros. Semi-flatness of the metric is part of a large conjectural picture by Gaiotto, Moore and Neitzke [GMN1, GMN2]. The semi-flatness refers to the Hitchin fibration $\mathcal{M}_{\mathrm{SL}}(d) \to \mathcal{Q}$ where $\mathcal{Q}$ is the space of holomorphic quadratic differentials and the map is given by taking the determinant of the Higgs field. The basic idea is that the metric on the fibres of the fibration is flat. The ideas are not fully developed but in [MSWW1], but in the review [MSWW2] the following idea is presented: A family of limiting configurations $(A_\infty(s), \Phi_\infty(s))$ associated to a curve of holomorphic quadratic differentials $q(s)$ can be perturbed (changing gauge) to a family of solutions $(A(s), t\Phi(s))$ of the Higgs bundle equations (1) with t sufficiently large (see Section 7). In a suitable gauge the derivative of this family with respect to s is a vertical tangent vector ot $\mathcal{M}_{\mathrm{SL}}(d)$ with respect to the Hitchin fibration. The hyperkähler metric can then be evaluated on this tangent vector, and seen to be an exponentially small correction to the vertical part of the semi-flat metric as $t \to \infty$. The horizontal tangent vectors can be obtained by varying the holomorphic quadratic differential giving a comparison between the horizontal parts of the metric.

It should be noted that the full metric on $\mathcal{M}_{\mathrm{SL}}(d)$ is not semi-flat but has correction terms coming from the fibres of the Hitchin fibration where the determinant of the Higgs field has non-simple zeros.

The focus of this exposition will be on the actual content of [MSWW1], that is the new proof of the Hitchin–Simpson theorem. We will not discuss the implications of the paper any further.

2 Exact solutions of the anti-self-dual equations and dimensional reductions

The main content of [MSWW1] concerns constructing solutions of the Higgs bundle equations. The Higgs bundle equations are reductions of the Anti-Self-Duality (ASD) equations in four real dimensions to two. Before we go into the details of how to construct solutions to the Higgs bundle equations, let us go a bit back in time and see how exact solutions of the ASD equations and its reduction to three dimensions were constructed.

2.1 Instantons

On a trivial bundle on $\mathbb{R}^4$ consider a connection A described by four skew Hermitian matrices $A_i(x)$ and covariant derivatives $\nabla_i = \frac{\partial}{\partial x_i} + A_i(x)$. The ASD equation is

$$* F_A = -F_A \tag{2}$$

where $*$ is the Hodge-star operator. We are looking for solutions to (2) under the boundary condition that the Yang–Mills action is finite

$$\int_{\mathbb{R}^4} \|F_A\|^2 = 8\pi^2 k + \int_{\mathbb{R}^4} \|F_A + *F_A\|^2 < \infty,$$

where k is an integer called the charge. Solutions to (2) are called *instantons*. One way to construct instantons on $\mathbb{R}^4$ is by the ADHM-construction [ADHM], but there are also other ways. If we want a charge 1 SU(2)-instanton which is also SO(4)-invariant then

$$A = \mathrm{Im}\left(\frac{x d\bar{x}}{1+|x|^2}\right),$$

where $x = x_0 + ix_1 + jx_2 + kx_3$ is a quaternion, is an instanton located at 0. If we take k such connections each centered at a different point, a_i, and the a_i's are sufficiently far apart then

$$A = A_1 + \cdots + A_k$$

is an approximate solution to equation (2). To get an exact solution one can use Taubes' grafting construction, [T].

2.2 Monopoles

In three dimensions we consider connections on a flat bundle on $\mathbb{R}^3$. In the same way as above we describe the connection in terms of three skew-Hermitian matrices $A_i(x)$ and now a Higgs field φ which is also skew-Hermitian. The ASD equations reduce to three dimensions by requiring translation invariance in the fourth component. These new equations are called *monopole* or *Bogomolny equations*

$$F_A = *\nabla\varphi. \tag{3}$$

The translation invariance essentially identifies A_4 and φ. Again the boundary condition is a requirement for finite Yang–Mills action

$$\int_{\mathbb{R}^3} \|F_A\|^2 + \|\nabla\varphi\|^2 = 4\pi k + \int_{\mathbb{R}^3} \|F_A - *\nabla\varphi\| < \infty$$

where k is an integer called the charge. Hitchin constructed the moduli space of monopoles, [H1], but if we again look for a charge 1 SU(2)-monopole which is also SO(3) invariant then

$$A = \left(\frac{1}{\sinh r} - \frac{1}{r}\right)\frac{1}{r} x \times dx \quad \text{and} \quad \varphi = \left(\frac{1}{\tanh r} - \frac{1}{r}\right)\frac{1}{r} x,$$

where $x = ix_1 + jx_2 + kx_3$, is a solution centered at 0. Again we can take k copies of this solution located at points a_i sufficiently far apart giving an approximate monopole

$$A = A_1 + \cdots + A_k.$$

We can again use Taubes' grafting procedure to obtain an exact solution, [JT].

2.3 Higgs bundles

The Higgs bundle equations on $\mathbb{R}^2$ are (like the monopole equations) dimensional reductions of the ASD equations by requiring solutions to be translation invariant in two directions. The connection matrices in these directions, say A_3, A_4, are turned into Higgs fields φ_1, φ_2 which define the usual Higgs field $\Phi = \frac{1}{2}(\varphi_1 + i\varphi_2)dz$, and the equations are

$$F_A + [\Phi, \Phi^*] = 0 \quad \text{and} \quad \bar{\partial}_A \Phi = 0.$$

The boundary condition of finite Yang–Mills action from the previous two cases is vacuous as the Yang–Mills functional is

$$\int_{\mathbb{R}^2} \|F_A + [\Phi, \Phi^*]\|^2 < \infty$$

with a minimum value of zero. In physics terms the connections with minimal action are, according to the minimal action principle, the physically interesting ones. From the action functional one derives the Euler–Lagrange equations giving the equations of motion in this physical theory. But as the minimum is zero the paths in this physical theory are just constants. This is the reason why physicists initially disregarded these equations, because there were not any immediately interesting physics, [L], now we of course know that Higgs bundles are indeed physically very interesting albeit in a physical theory which did not exist until 2007 with Kapustin and Witten, [KW].

If we look for SU(2)-Higgs bundles with extra SO(2)-symmetry they can be seen as solutions to the ASD equation invariant under $\mathrm{SO}(2) \times \mathbb{R}^2$. Such solutions can be found by essentially solving the Painlevé III equation, [MW],

$$(x\partial_x)^2\psi = \frac{1}{2}x^2 \sinh(2\psi). \tag{4}$$

As the Higgs bundle equations have the nice property of being conformally invariant they can be defined on a compact Riemann surface. We cannot immediately pursue the same strategy as above by taking local solutions positioned sufficiently far apart by the compactness. Nevertheless this is roughly what is done in [MSWW1] where the problem of compactness is overcome by requiring the Higgs field to have sufficiently large L^2-norm.

3 Main theorem

The main result in [MSWW1] is a constructive proof of the Hitchin–Simpson theorem for a certain class of Higgs bundles.

Theorem 3.1. *Let E be a fixed complex vector bundle of rank two with a fixed holomorphic structure on $\det E$, and assume $(\bar\partial, \Phi)$ is a Higgs bundle with $\det\Phi$ having simple zeros. Then there is a Hermitian metric on E such that when t is sufficiently large there is a solution (A_t, Φ_t) to*

$$F_A^\perp + t^2[\Phi, \Phi^*] = 0 \quad \textit{and} \quad \bar\partial_A \Phi = 0$$

in the $\Gamma(\mathrm{SL}(E))$-gauge orbit of $(A, t\Phi)$ where A is the Chern connection of $\bar\partial$.

If we compare this formulation to Theorem 1.1 the polystability is replaced by $\det\Phi$ having simple zeros as this implies stability of $(\bar\partial, \Phi)$ when the rank of E is two.

3.1 Strategy for the proof

The proof consists of several parts and in this section we will give a brief overview of the structure of the proof and the strategy chosen.

1. A *limiting configuration* is a singular pair (A_∞, Φ_∞) of a unitary connection and a Higgs field both with standardized behaviour at the singularities, D, solving a decoupled version of the Higgs bundle equations away from D, i.e. $F(A_\infty) = 0$ and $[\Phi_\infty, \Phi_\infty^*] = 0$, see Section 5 for the definition.

2. For $(\bar{\partial}, \Phi)$ a rank two Higgs bundle of odd degree with $\det \Phi$ having simple zeros, D, there is a Hermitian metric H such that (A, Φ) is complex gauge equivalent on $X^\times = X \setminus D$ to a limiting configuration where A is the Chern connection.

3. Construct a local family of smooth SO(2)-invariant solutions to the Higgs bundle equations converging to the singular behaviour defined by the limiting configurations.

4. Glue this family to the limiting configuration to obtain an approximate solution to the Higgs bundle equations on all of X.

5. Perturb the approximate solutions to obtain an exact solution which is complex gauge equivalent to the pair we started with.

The constructive aspect of this strategy will give control of the shapes of the solutions and the gauge transformations. This is important to determine the asymptotics of the L^2-metric on the moduli space.

4 Local solutions

In this section we construct local model solutions on $\mathbb{R}^2$ called *fiducial* solutions. First let us consider a singular limiting fiducial solution $(A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}})$.

Let B be an open disk centered at 0 and denote $B^\times = B \setminus \{0\}$. Let E be a complex vector bundle on B and fix a Hermitian metric on E. Choose a unitary frame trivialising E on $B^\times$. In this frame define

$$A_\infty^{\text{fid}} = \frac{1}{8}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right) \quad \text{and} \quad \Phi_\infty^{\text{fid}} = \begin{pmatrix}0 & r^{1/2}\\ r^{1/2}e^{i\theta} & 0\end{pmatrix} dz$$

where Φ_∞^{fid} is specified in polar coordinates. Notice that the connection A_∞^{fid} is singular at 0; Φ_∞^{fid} is continuous at 0 and otherwise smooth; Φ_∞^{fid} is normal and $\det \Phi_\infty^{\text{fid}} = -z dz^2$ has a simple zero.

Lemma 4.1. *The connection A_∞^{fid} is flat on $B^\times$ and Φ_∞^{fid} is holomorphic with respect to A_∞^{fid}, so especially they satisfy the Higgs bundle equations on $B^\times$.*

Secondly we construct a family of smooth solutions on B to

$$0 = \mathcal{H}_t(A, \Phi) = (F_A + t^2[\Phi, \Phi^*], \bar{\partial}_A \Phi) \quad t > 0 \tag{5}$$

which converge to $(A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}})$ as $t \to \infty$.

Lemma 4.2. *The pair $(A_t^{\text{fid}}, \Phi_t^{\text{fid}})$ defined by*

$$A_t^{\text{fid}} = f_t(r)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right) \quad \textit{and} \quad \Phi_t^{\text{fid}} = \begin{pmatrix}0 & r^{1/2}e^{h_t(r)}\\ r^{1/2}e^{i\theta}e^{-h_t(r)} & 0\end{pmatrix} dz$$

solves (5) *where*

$$f_t(r) = \frac{1}{8} + \frac{1}{4} r \partial_r h_t \quad \textit{and} \quad h_t(r) = \psi\left(\frac{8}{3} t r^{3/2}\right)$$

with ψ *a solution to the Painlevé III equation* (4).

It turns out that if we want exponential decay of ψ and the right behaviour at 0 then there is a unique solution to (4). As we argued in Section 2.3 the Painlevé III equation had to enter the picture somewhere, and this is how.

Lemma 4.3. *The pair* $(A_t^{\mathrm{fid}}, \Phi_t^{\mathrm{fid}})$ *is smooth at* 0 *and converge exponentially in* t*, uniformly in* C^∞ *on any exterior region* $r \geq r_0 > 0$ *to* $(A_\infty^{\mathrm{fid}}, \Phi_\infty^{\mathrm{fid}})$.

In the frame fixed above define

$$A_0 = 0 \quad \text{and} \quad \Phi_0 = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} dz.$$

Lemma 4.4. *For every* $t > 0$ *the pair* $(A_t^{\mathrm{fid}}, \Phi_t^{\mathrm{fid}})$ *is complex gauge equivalent to* (A_0, Φ_0) *on* B*. The limiting fiducial pair* $(A_\infty^{\mathrm{fid}}, \Phi_\infty^{\mathrm{fid}})$ *is complex gauge equivalent using a singular gauge transformation to* (A_0, Φ_0) *on* $B^\times$.

5 Limiting configuration

As mentioned in Section 3.1 the first part of the proof is to find a so-called *limiting configuration* which is complex gauge equivalent on $X^\times$ to an initial pair (A, Φ).

Definition 5.1. *Let* E *be a Hermitian complex vector bundle on* X *and* $D = p_1 + \cdots + p_{4g-4}$*. A limiting configuration is a Higgs pair on* $X^\times = X \setminus D$*,* (A_∞, Φ_∞) *satisfying the decoupled Higgs bundle equations*

$$F_{A_\infty}^\perp = 0 \quad [\Phi_\infty, \Phi_\infty^*] = 0 \quad \bar{\partial}_{A_\infty} \Phi_\infty = 0$$

and which agrees with $(A_\infty^{\mathrm{fid}}, \Phi_\infty^{\mathrm{fid}})$ *near each point of* D *with respect to some holomorphic coordinate system and unitary frame for* E.

The trace-free condition on the curvature of A_∞ is because we require the induced connection on $\det E$ to be fixed.

The main theorem about limiting configurations is the following.

Theorem 5.2. *Let* $(\bar{\partial}, \Phi)$ *be a Higgs bundle of rank two, odd degree, and with simple zeros of* $\det \Phi$*. Then there is a Hermitian metric so that if* A *is the associated Chern connection then the pair* (A, Φ) *is complex gauge equivalent on* $X^\times$ *to a limiting configuration* (A_∞, Φ_∞).

There is essentially two steps in the proof of Theorem 5.2:

1. Make Φ normal away from D.
2. Flatten A away from D without changing the normality property of Φ.

5.1 Normal form for Higgs bundles

First let us mention a convenient lemma.

Lemma 5.3. *In a neighbourhood of any simple zero of* $\det\Phi$ *there is a complex coordinate* z *and a local holomorphic frame of* E *such that*

$$\Phi = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} dz.$$

If $(\bar\partial, \Phi)$ is a Higgs bundle with simple Higgs field then in the neighbourhood and frame from Lemma 5.3 we define a Hermitian metric H_0 by declaring this frame to be unitary. We do this for every point of D and finally extend H_0 arbitrarily to the rest of X. Notice that if A is the Chern connection of $(H_0, \bar\partial)$ then A vanish on the neighbourhoods and in the frames fixed from Lemma 5.3.

From Lemma 4.4 we can find a local singular complex gauge transform changing (A, Φ) around the points in D to fiducial limiting solutions. It is actually possible, [MSWW1, Lemma 4.3], to extend these local complex gauge transforms to complex gauge tranforms defined on $X^\times$. That is, there is a $g \in \Gamma(X^\times, \mathrm{SL}(E))$ such that $\Phi^g = g^{-1}\Phi g$ is normal on $X^\times$.

5.2 Remove trace-free part of curvature

In this section we will discuss how to gauge away the trace-free part of the curvature without changing the normal-form of Φ. We want to find $g \in \Gamma(X^\times, \mathrm{SL}(E))$ such that

$$F^\perp_{A^g} = 0 \quad \text{and} \quad g^{-1}\Phi g = \Phi.$$

We therefore define the holomorphic line bundle

$$L^{\mathbb{C}}_\Phi = \{\gamma \in \mathfrak{sl}(E) \,|\, [\gamma, \Phi] = 0\}$$

of infinitesimal stabilisers of Φ and consider the real line bundles $L_\Phi = L^{\mathbb{C}}_\Phi \cap \mathfrak{su}(E)$, iL_Φ of skew-Hermitian and Hermitian elements. If we then take $g = \exp(\gamma)$ with $\gamma \in iL_\Phi$ then changing gauge with g does not change the normal form of Φ. We then want to solve

$$0 = F^\perp_{A^g} = g^{-1}(F^\perp_A + \bar\partial_A(gg^*\partial_A(gg^*)^{-1}))g = g^{-1}(F^\perp_A - 2\bar\partial_A\partial_A\gamma)g,$$

where the last equality follows as γ is Hermitian. Solving this equation is equivalent to solving the Poisson equation

$$\Delta_A \gamma = i\Lambda F_A^{\perp}$$

where Λ is contraction with a chosen Kähler form on X.

The theorem is proved if we can find a solution to the Poisson equation. The problem is however that A has a simple pole at the zeros of $\det \Phi$ and even though Δ_A is an elliptic operator we cannot just use standard theory to solve this equation. Because of the singularity we have to use weighted Sobolev spaces and appeal to the theory of conical elliptic operators. As this is just an exposition of the ideas of the paper we won't go into the details needed to solve the Poisson equation.

The proof of Theorem 5.2 is strictly gauge theoretic. There is however a different proof, suggested by Hitchin, which is completely geometric. The proof involves spectral data for $(\bar{\partial}, \Phi)$ and will be the topic of Section 9.

6 Approximate solutions

Now that we have a gauge transform g_∞ taking the initial pair (A, Φ) to a limiting configuration (A_∞, Φ_∞) on $X^\times$, we want to smoothen this at the points of D to get a family of Higgs pairs converging to (A_∞, Φ_∞) which are defined on X and are approximate solutions to the Higgs bundle equations.

Define $X^{int} = \cup_{p \in D} B_1^\times(p)$ and $X^{ext} = X \setminus \bar{X}^{int}$ and assume the limiting configuration is fiducial on X^{int}. Define furthermore a complex gauge transformation $g_t = \exp(\gamma_t)$ where

$$\gamma_t = \begin{pmatrix} -\frac{1}{2}h_t & 0 \\ 0 & \frac{1}{2}h_t \end{pmatrix}$$

where h_t are the solutions to the Painlevé III equation as in Lemma 4.2. Then by definition on X^{int} we have

$$(A_t^{\mathrm{fid}}, \Phi_t^{\mathrm{fid}}) = (A_\infty^{\mathrm{fid}}, \Phi_\infty^{\mathrm{fid}})^{g_t}.$$

Choose a smooth cut-off function $\chi : X \to [0,1]$ with support in X^{int} and being constant 1 on $\cup_{p \in D} B_{\frac{1}{2}}(p)$, then the gauge transformation $g_t^{\mathrm{app}} = \exp(\chi \gamma_t)$ is a family of smooth complex gauge transformations on X with $g_t^{\mathrm{app}} = g_t$ on $\cup_{p \in D} B_{\frac{1}{2}}(p)$ and $g_t^{\mathrm{app}} = Id$ on X^{ext}. Using this gauge transform we "glue in" the smooth fiducial solutions to the limiting configuration in a small neighbourhood of the points of D.

To summarise: If (A, D) is the initial pair, then we use a complex gauge transform g_∞ on $X^\times$ to change to a limiting configuration (A_∞, Φ_∞) and then g_t^{app} to get the pair $(A_t^{\mathrm{app}}, \Phi_t^{\mathrm{app}})$. A priori the last pair is only defined on $X^\times$, but by considering the local behaviour of $g_\infty g_t^{\mathrm{app}}$ at $p \in D$ it turns out that (A, Φ) is complex gauge equivalent to $(A_t^{\mathrm{app}}, \Phi_t^{\mathrm{app}})$ on X.

The pair $(A_t^{\mathrm{app}}, \Phi_t^{\mathrm{app}})$ is an approximate solution to the rescaled Higgs bundle equations in the sense that there are constants $C, \delta > 0$ independent of t, such that for $t \gg 1$

$$\|F^\perp_{A_t^{\mathrm{app}}} + t^2[\Phi_t^{\mathrm{app}}, (\Phi_t^{\mathrm{app}})^*]\|_{L^2} \leq Ce^{-\delta t}.$$

To obtain an exact solution when t is large, we have to perturb this solution slightly. We perturb by once again changing the gauge.

7 Perturbation to exact solutions

We again gauge by $\exp(\gamma)$ where γ is a section of iL_Φ. This means that we want to solve

$$\begin{aligned} 0 &= F^\perp_{(A_t^{\mathrm{app}})^{\exp(\gamma)}} + t^2[(\Phi_t^{\mathrm{app}})^{\exp(\gamma)}, ((\Phi_t^{\mathrm{app}})^{\exp(\gamma)})^*] \\ &= F^\perp_{A_t^{\mathrm{app}}} + t^2[\Phi_t^{\mathrm{app}}, (\Phi_t^{\mathrm{app}})^*] + L_t\gamma + Q_t\gamma, \end{aligned} \tag{6}$$

where

$$L_t\gamma = \Delta_{A_t^{\mathrm{app}}}\gamma + t^2 M_{\Phi_t^{\mathrm{app}}}\gamma \quad \text{and} \quad M_\varphi\gamma = 2([\varphi^*, [\varphi, \gamma]] + [\varphi, [\varphi^*, \gamma]])$$

and $Q_t\gamma$ being the remaning terms.

If we set up the problem correctly and use the right Sobolev spaces the operator L_t is invertible. Then we have a solution to (6) if and only if

$$\gamma = -L_t^{-1}(F^\perp_{A_t^{\mathrm{app}}} + t^2[\Phi_t^{\mathrm{app}}, (\Phi_t^{\mathrm{app}})^*] + Q_t\gamma) = T(\gamma)$$

that is if and only if γ is a fixed point of T.

To find a fixed point it is enough to prove that T is a contraction mapping of a ball into itself. This will follow if we have the proper estimates on the operator norm of L_t^{-1} and Q_t. This is a quite technical part of the paper and we refer the interested reader to Section 6 in [MSWW1] for the details.

8 Summary

What did we actually prove? If E is a complex vector bundle of rank two and $(\bar\partial, \Phi)$ a Higgs bundle structure on E with $\det\Phi$ having only simple zeros, then we gave a construction of a Hermitian metric suited to the Higgs field Φ and with this Hermitian metric fixed we constructed (for t sufficiently large) a gauge transform g_t, such that if A is the Chern connection, then $(A_t, \Phi_t) = (A, \Phi)^{g_t}$ is a solution to

$$\bar\partial_{A_t}\Phi_t = 0 \quad \text{and} \quad F^\perp_{A_t} + t^2[\Phi_t, \Phi_t^*] = 0.$$

This is the difficult direction in Theorem 1.1.

9 Alternative construction of limiting configurations

Essential to the proof is the construction of limiting configurations. The presentation in this note and in Section 4 of [MSWW1] is analytically very heavy. As it turns out limiting configurations are actually rather natural objects to study as the following alternative construction proposed by Hitchin suggests.

Since we require the Higgs bundles to have fixed holomorphic structure on $\det E$ we will just focus on the $\mathrm{SL}(2,\mathbb{C})$ case for simplicity. The construction works for the odd degree cases as well.

Let $q \in H^0(\Sigma, K^2)$ have simple zeros, and let $\pi : S \to X$ be the $2:1$-covering of X branched at the $4g-4$ zeros of q defined by the square root of q (i.e. S is the curve in the total space of the canonical bundle defined by the equation $\eta^2 = q$ where η is the tautological section of the canonical bundle pulled back along itself). Finally let $\sigma : S \to S$ be the involution permuting the sheets.

Proposition 9.1. *Let U be a line bundle on S with the property that $\sigma^* U \otimes U \simeq \pi^*(K)$ then there is a Hermitian metric on the holomorphic vector bundle*

$$E = \pi_*(U \oplus \sigma^* U)^\sigma$$

such that the Chern connection and the Higgs field

$$\pi_* \begin{pmatrix} \sqrt{q} & 0 \\ 0 & -\sqrt{q} \end{pmatrix}$$

is a limiting configuration on Σ with singularities at the zeros of q.

Proof. Let L be a *flat* line bundle on S with the property that $\sigma^* L \simeq L^*$ (i.e. L is in the Prym-variety of S). Then $L \oplus L^*$ is a flat σ-invariant bundle. Then the push forward of the σ-invariant sections of $L \oplus L^*$ is a rank two holomorphic vector bundle on Σ with determinant K^{-1}. Moreover E has a Hermitian metric with a certain behaviour at the zeros of q. The Hermitian metric comes as part of a parabolic structure on E supported at the zeros of q. At each of the zeros of q there is a full flag of the fibre of E with weights 0 and $\frac{1}{2}$. Therefore the Hermitian metric behaves as

$$\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$$

at the zeros of q. Furthermore as $L \oplus L^*$ is flat, the Chern connection on E is flat as well – but only away from the branch points.

The Hermitian metric of a limiting configuration behaves as

$$\begin{pmatrix} r^{-1/2} & 0 \\ 0 & r^{1/2} \end{pmatrix}$$

at the zeros of q which is different to the behaviour of the parabolic metric.

We also need $\det E \simeq \mathcal{O}$. If L is twisted by $\pi^*(K^{1/2})$ where $K^{1/2}$ is a chosen square root of K, then

$$\pi_*(L \otimes \pi^* K^{1/2} \oplus \sigma^*(L \otimes \pi^* K^{1/2}))^\sigma = \pi_*((L \oplus L^*) \otimes \pi^* K^{1/2})^\sigma = E \otimes K^{1/2}.$$

Certainly $\det(E \otimes K^{1/2}) \simeq \mathcal{O}$ and to get a Hermitian metric on $E \otimes K^{1/2}$ we can twist the parabolic metric on E by a section of $K^{-1/2} \otimes \bar{K}^{-1/2}$. As q is a quadratic differential $(q\bar{q})^{-1/4}$ is exactly such a section. If h is the parabolic metric on E then as q has only simple zeros the behaviour at the branch points is

$$h(q\bar{q})^{-1/4} \sim \begin{pmatrix} r^{-1/2} & 0 \\ 0 & r^{1/2} \end{pmatrix}.$$

We can therefore obtain the bundle and connection of a limiting configuration by considering the push forward of the σ-invariant sections of a line bundle $U = L \otimes \pi^* K^{1/2}$ satisfying

$$\sigma^* U = \sigma^*(L \otimes \pi^* K^{1/2}) = L^* \otimes \pi^* K^{1/2} = U^* \otimes \pi^* K.$$

To get a limiting configuration we need a Higgs field which is normal away from the branch points. The square root $\alpha = \sqrt{q}$ is a well defined holomorphic differential on S, $\alpha \in H^0(S, K_S)$. The homomorphism

$$\varphi = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} : L \oplus L^* \to (L \oplus L^*) \otimes K_S$$

is normal on S and as it is σ-invariant it thus induces a Higgs field Φ' on E normal away from the branch points. $\square$

9.1 Limiting configuration associated to a Higgs bundle

Let (V, Φ) be an $\mathrm{SL}(2, \mathbb{C})$-Higgs bundle, i.e. $\det V \simeq \mathcal{O}$ and $\operatorname{Tr} \Phi = 0$. Assume furthermore that $-q = \det \Phi \in H^0(K^2)$ has simple zeros, $D = p_1 + \cdots + p_{4g-4}$.

Proposition 9.2. *A limiting configuration for (V, Φ) as above is (E, Φ') with*

$$E = \pi_*(U \oplus \sigma^* U)^\sigma \quad \text{and} \quad \Phi' = \pi_* \begin{pmatrix} \sqrt{q} & 0 \\ 0 & -\sqrt{q} \end{pmatrix}$$

where $\pi : S \to \Sigma$ is the spectral curve of (V, Φ) and U is a line bundle satisfying $\pi_* U = V$ and E is equipped with the Hermitian metric from Proposition 9.1.

Proof. The spectral curve S for (V, Φ) is the $2:1$ branched cover of Σ defined by q. Let $\sigma : S \to S$ be the involution changing the sheets of the covering.

Pulling back $V \xrightarrow{\Phi} V \otimes K$ to S gives the following exact sequence

$$0 \to U \otimes \pi^* K^{-1} \to \pi^* V \xrightarrow{\eta - \Phi} \pi^*(V \otimes K) \to U \otimes \pi^* K \to 0,$$

where U is a line bundle such that $\pi_* U = V$. By considering the norm map, it follows that

$$\sigma^* U \simeq U^* \otimes \pi^* K.$$

From Proposition 9.1 we get a limiting configuration, (E, Φ'), on Σ with the required properties. A gauge transformation between (V, Φ) and (E, Φ') exists as on S, $\pm\sqrt{q}$ are the eigenvalues of $\pi^*\Phi$ on $S \setminus \pi^* D$ and the flat bundle $L \oplus L^*$ on $S \setminus \pi^* D$ is an eigenspace decomposition of V. $\square$

This gives a different proof to Theorem 4.1 of [MSWW1].

It is interesting to see parabolic Higgs bundles find their way into the description of the hyperkähler metric on the moduli space of ordinary Higgs bundles. It would be interesting to see if their presence could be more actively used in the investigation of asymptotical properties of the metric.

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