MAXIMAL REPRESENTATIONS - FRAGMENTS OF SOLUTIONS

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2. Topology of $\mathrm{Homeo}^+(S^1)$

17. Consider the map

$$\begin{array}{rcl}\tilde{G} \times [0,1] & \to & \tilde{G} \\ (f,t) & \mapsto & (x \mapsto tx + (1-t)f(x)),\end{array}$$

it is well defined according to question 16. It is a continuous retraction from $\tilde{G}$ onto $\{id_{\mathbf{R}}\}$, hence $\tilde{G}$ is contractible.

21. According to the polar decomposition, $\mathrm{PSL}(2,\mathbf{R})$ deformation retracts onto $\mathrm{PSO}(S)$, hence $\pi_1\mathrm{PSL}(2,\mathbf{R}) \simeq \mathbf{Z}$.

The unique connected k-cover $\mathrm{PSL}(2,\mathbf{R})_{(k)}$ of $\mathrm{PSL}(2,\mathbf{R})$ comes from the unique subgroup of $\pi_1\mathrm{PSL}(2,\mathbf{R}) \simeq \mathbf{Z}$ of index k, namely $k\mathbf{Z}$.

If we consider the connected k-cover of the circle $p_k : S^1 \to S^1$ (the multiplication by k), then $\mathrm{PSL}(2,\mathbf{R})_{(k)}$ can be thought of as the group of all lifts of elements of $\mathrm{PSL}(2,\mathbf{R}) \subset \tilde{G}$ under p_k. Or if we denote by r_k the rotation of S^1 of order k, then $\mathrm{PSL}(2,\mathbf{R})_{(k)}$ can be realized as the subgroup of $\mathrm{PSL}(2,\mathbf{R})$ which commutes with r_k. These subgroups are not conjugated in $\mathrm{Homeo}^+(S^1)$ as the center of $\mathrm{PSL}(2,\mathbf{R})_{(k)}$ has order k.

23. Lemma about $\underline{m}$ and $\overline{m}$ stated in the first lecture.

- $0 \leq \overline{m}(f) - \underline{m}(f)$ is immediate. Adding to f a real constant does not change the quantity $\overline{m}(f) - \underline{m}(f)$, so we may assume $f(0) = 0$. Let $x, y \in [0,1)$ such that $\overline{m}(f) = f(x) - x$ and $\underline{m}(f) = f(y) - y$. Then if $y \leq x$, $f(y) - y \geq -x$ so $\overline{m}(f) - \underline{m}(f) \leq f(x) < 1$; and if $y \geq x$, $f(y) - y > f(x) - 1$ so $\overline{m}(f) - \underline{m}(f) < 1 - x \leq 1$.
- $\underline{m}(f^{-1}) = \min\{f^{-1}(f(y)) - f(y) | x = f(y) \in \mathbf{R}\} = -\overline{m}(f)$.
- Assume that $\underline{m}[f,g] \geq 1$, that is $\forall x \in \mathbf{R}, fgf^{-1}g^{-1}(x) - x \geq 1$. Hence $\forall x \in \mathbf{R}, f(g(x)) \geq g(f(x)) + 1$. Up to adding an integer to f, we may assume that $\underline{m}(f) \in [0,1)$: in particular $\forall x \in \mathbf{R}, f(x) \geq x$ so $f(g(x)) \geq g(x) + 1$, hence $\forall y \in \mathbf{R}, f(y)_y \geq 1$ so $\underline{m}(f) \geq 1$: contradiction. So $\underline{m}[f,g] < 1$.

4. Maximal Representations

31. If $\tau : G \to H$ is a morphism of topological groups, then $I_H(\tau \circ \rho) = \tau_*(I_G(\rho))$, where $\tau_* : \pi_1 G \to \pi_1 H$ is the induced morphism. In this case, we just need to show that $\tau_* : \pi_1\mathrm{SL}(2,\mathbf{R}) \simeq \mathbf{Z} \to \pi_1\mathrm{Sp}(2n,\mathbf{R}) \simeq \mathbf{Z}$ is the multiplication by n. A generator of $\pi_1\mathrm{SL}(2,\mathbf{R})$ is the classe of the loop $c : \theta \in [0,2\pi] \mapsto \left(\begin{smallmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{smallmatrix}\right)$, and its image under τ is $\tau \circ c : \theta \in [0,2\pi] \mapsto \left(\begin{smallmatrix}\cos\theta \mathrm{I}_n & -\sin\theta \mathrm{I}_n \\ \sin\theta \mathrm{I}_n & \cos\theta \mathrm{I}_n\end{smallmatrix}\right)$. But one generator of $\pi_1\mathrm{Sp}(2n,\mathbf{R})$ is the class of the loop $c' : \theta \in [0,2\pi] \mapsto \begin{pmatrix}\cos\theta & 0 & -\sin\theta & 0 \\ 0 & \mathrm{I}_{n-1} & 0 & \mathrm{I}_{n-1} \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \mathrm{I}_{n-1} & 0 & \mathrm{I}_{n-1}.\end{pmatrix}$, so the homotopy class of $\tau \circ c$ is n times the homotopty class of c'. Hence τ_* is the multiplication by n.

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Note that a representation in $\mathrm{SL}(2,\mathbf{R})$ is maximal if its Euler number is $g-1$. And when we look at representations in $\mathrm{PSL}(2,\mathbf{R})$, then the maximal value is $2g-2$, since $\mathrm{SL}(2,\mathbf{R})$ is the connected 2-cover of $\mathrm{PSL}(2,\mathbf{R})$.

32. τ is defined as follows. Consider the vector space V of homogeneous polynomials in X, Y of degree $2n-1$. The canonical basis is $X^{2n-1}, X^{2n-2}Y, \ldots, Y^{2n-1}$. $\mathrm{SL}(2,R)$ acts on V by the following : if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $g.X^pY^q = (aX+bY)^p(cX+dY)^q$. This defines an irreducible representation τ of $\mathrm{SL}(2,\mathbf{R})$ in $\mathrm{GL}(V) = \mathrm{GL}(2n,\mathbf{R})$.

Consider the following symplectic form ω on V : if $(p,q) \neq (q',p')$, then $\omega(X^pY^q, X^{p'}Y^{q'}) = 0$. And $\omega(X^pY^q, X^qY^p) = 1$ if $p > q$, and -1 if $p < q$. The matrix of ω in the canonical basis is the canonical matrix $\left(\begin{smallmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{smallmatrix}\right)$.

It is clear that ω is invariant under diagonal and triangular matrices in $\mathrm{SL}(2,\mathbf{R})$, hence ω is $\mathrm{SL}(2,\mathbf{R})$-invariant.

Hence $\tau(\mathrm{SL}(2,\mathbf{R})) \subset \mathrm{Sp}(2n,\mathbf{R})$.

For $\theta \in \mathbf{R}$, denote $r_\theta \in \mathrm{SL}(2,\mathbf{R})$ the rotation of angle θ. Then we have $r_\theta \cdot X + iY = e^{i\theta}(X+iY)$ and $r_\theta \cdot X - iY = e^{-i\theta}(X-iY)$. Hence the complex eigenvectors of the action of r_θ on the complexified vector space $V \otimes \mathbf{C}$ are, for all $p \in [0, 2n-1]$:

$$r_\theta \cdot (X+iY)^p(X-iY)^{2n-1-p} = e^{i(2p-2n+1)\theta}(X+iY)^p(X-iY)^{2n-1-p}.$$

When we now consider the n invariant real 2-planes, they are, for all $p \in [0, n-1]$, spanned by the real and imaginary parts of $Z_p = (X+iY)^p(X-iY)^{2n-1-p}$. But we remark that the sign of $\omega(ReZ_p, ImZ_p)$ is the same as $(-1)^{p+1}$: the orientation of the loops alternate. Hence the image by τ of the standard loop c of $\mathrm{SL}(2,\mathbf{R})$ is composed of n loops of total angles $(-1)^{p+1}(2p-2n+1)$. Hence τ_* is equal to the multiplication by

$$\sum_{p=0}^{n}(-1)^{p+1}(2p-2n+1) = n.$$

So, as in the previous exercise, we get that $\tau \circ \rho$ is maximal.