INTRODUCTION TO (G, X)-STRUCTURES
PROBLEM SET SOLUTIONS

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Problem Set 1

Problem 1. Denote the natural projection by $\pi : X \to \Gamma\backslash X$. Let $x, y \in X$ such that $\pi(x) \neq \pi(y)$. We are looking for open sets $U, V \subset \Gamma\backslash X$ such that $\pi(x) \in U, \pi(y) \in V$ and $U \cap V = \emptyset$.

It suffices to find open sets $U, V \subset X$ such that for all γ_1, γ_2 we have $\gamma_1 U \cap \gamma_2 V = \emptyset$ and $x \in U, y \in V$. Let B be a small ball around x. Since Γ acts properly discontinuously, only finitely many sets of the form γB have $\gamma B \cap B \neq \emptyset$. If any of them contain x, make B smaller. Set $U = B \backslash \cup_\Gamma \gamma B$. Define V likewise, but also cut out translates of U (shrinking it a bit more if necessary).

Problem 2. We must also assume g has finite order. Otherwise, the identity matrix is a counter-example. We break into a few cases:

Suppose g has two real eigenvalues with "opposite sign". Then we may restrict to the corresponding plane and just look at the action of the corresponding diagonal 2-by-2 matrix. Let K be the unit circle. Then $g^n K \cap K \neq \emptyset$ for all n.

Suppose g has two complex eigenvalues with "opposite sign". Then we may restrict to $\mathbb{C}^2$ and let K be the unit sphere. Again, $g^n K \cap K \neq \emptyset$ for all n.

Conversely, suppose g has all eigenvalues with norm more than 1. Let K be compact. Then it is bounded by spheres away from both 0 and ∞. This bounding annulus goes off to 0 or ∞ under the action of g, so the action is properly discontinuous.

Problem 3. We know that γ is homotopic to the trivial path. Decompose $[0, 1] \times [0, 1]$ as in the hint so that each square lies entirely in the domain of a single chart. We then know that going around a single square does not change the image of the point under the developing map. That means we can push γ past one square without changing $\text{dev}(\gamma(1))$. So then we can push past all the squares, and eventually γ will become the trivial path. We then have $\text{dev}(\gamma(1)) = \text{dev}(\gamma(0))$.

Problem 4. By the previous problem, we know that the developing map is well-defined on the universal cover of S^1 and is π_1-equivariant. That is, we have a local diffeomorphism $\mathbb{R} \to \mathbb{R}$ such that translating by 1 in $\mathbb{R}$ (a deck transformation) corresponds to a motion by $\text{hol}(g)$ (where $\pi_1(S^1) = \langle g \rangle$). Furthermore, $\text{hol}(g)$ is a similarity. In this case, that means it has the form $x \mapsto ax + b$ for some a, b.

Considering what $\text{hol}(\gamma)$ and $\text{dev}(R)$ might be yeilds two non-equivalent cases: $\text{dev}(\mathbb{R}) = R$ or $\text{dev}(\mathbb{R}) = (0, \infty)$. The first structure is complete, the second isn't.

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Problem 5. An affine manifold is geodesically complete if the exponential map is defined on the entire tangent space. Note that we have

$$\operatorname{dev} \exp = \exp D\operatorname{dev}$$

where the first exponential is on M, while the second is on $\mathbb{R}^n$. Since the developing map is a diffeomorphism, $D\operatorname{dev}$ is an isomorphism. Furthermore, the Euclidean exponential is onto.

Thus, the developing map is onto exactly if the exponential on M is defined everywhere.

Problem Set 2

Problem 1. Assume M does have a (G, X)-structure. Since M is compact, the developing map is a map from the universal cover $\widetilde{M}$ onto a connected component of X. We assume X is connected, so the developing map is onto. Since $\pi_1 M$ is finite, $\widetilde{M}$ is compact. But there's no continuous map from a compact set onto an unbounded one, a contradiction.

Problem 2. This problem caused a lot of discussion because there are at least two completely distinct ways of solving it. Either way, note that if such a structure does exist, it is Riemannian (see below), and therefore complete since the torus is compact. Let Γ be the image of the holonomy map, acting co-compactly on the Heisenberg group. Recall that the Heisenberg group is the group of upper-triangular 3x3 matrices with ones on the diagonal. A more geometric model is the space $\mathbb{C} \times \mathbb{R}$ with group structure

$$(z, t) * (z', t') = (z + z', t + t' + 2\operatorname{Im}(\overline{z}z')).$$

One can give the Heisenberg group a Riemannian metric by using the standard inner product at the origin and extending by left multiplication.

Algebraic Approach. The Borel density theorem implies that if Γ is a lattice subgroup of a connected semi-simple real algebraic Lie group G with no compact factors, then Γ is Zariski dense in G.

Our group Γ is a lattice in $SU(1) \ltimes \text{Heis}$. Taking a subgroup of finite index, we may assume Γ is a subgroup of Heis itself. Applying the Borel density theorem, we have that Γ is Zariski dense in Heis. In particular, any polynomial that is zero on Γ is zero on all of Heis. But Γ is abelian, and Heis isn't. This is a contradiction, so there is no Nil-structure on the torus.

Geometric Group Theory Approach. Let Γ be a finitely generated group with fixed generators. The word metric on Γ is defined by taking $d(\gamma, \gamma') = 1$ any time $\gamma^{-1}\gamma'$ is a generator of Γ. Recall that a quasi-isometry is a mapping f between metric spaces so that for some $L \geq 1, C \geq 0$ we have

$$-C + L^{-1} d(x, y) \leq d(fx, fy) \leq L d(x, y) + C.$$

It is easy to see that changing the generators of Γ induces a quasi-isometry between the word metrics.

To tell the difference between two groups, one can use the growth function. For fixed generators, define a function that assignes to n the number of elements of Γ that can be written using n or less generators. Under appropriate equivalence, the growth function is independent of the choice of generators, and furthemore is a quasi-isometry invariant. For example, the growth function of $\mathbb{Z}^3$ is (up to equivalence) n^3.

The Milnor-Svarc lemma states that if a group Γ acts properly discontinuously and co-compactly on a metric space X, then Γ is finitely generated and, with the word metric, quasi-isometric to X. Thus, two groups acting properly discontinuously and co-compactly on the same metric space have different growth functions.

Note now that the integer Heisenberg group (restrict Heis to integer entries) acts properly discontinuously and co-compactly on Heisenberg space, but has growth rate n^4. The last fact is easy to check since the integer Heisenberg group is given by the presentation $\langle x, y, t \ : \ [x,y] = 4t, [x,t] = [y,t] = 1\rangle$, and its elements are easy to normalize.

Thus, the dimension of $\mathbb{Z}^3$ is too small for it to act on the Heisenberg group, which is for many purposes 4-dimensional.

Problem 3. Recall that given a metric space, the Hausdorff distance between two compact subsets is the smallest number D such that the D-neighborhood of each set contains the other. Likewise, one can define a Hausdorff topology on the space of compact subsets of a topological space.

The key to this problem is to think of an ellipse as the set of "axis" vectors that define it (which makes sense for a generic ellipse). Two ellipses are close exactly when their axes are close. Now, each axis vector can either converge to zero, converge to a finite vector, or converge to infinity. For the ellipse this means, respectively, gaining a co-dimension, converging to an elliptical product component, or converging to an $\mathbb{R}$ product component.

In the notation, the discompactness of the sequence is the number of axes that converged to 0, d is the number of axes that converged to a finite vector, and l is the number of vectors that converged to infinity. (This counting ignores the possibility that axes become linearly dependent in the limit.)

Problem 4. We use the interpretation of discompactness digree from the previous problem. Namely, it is the number of axes that converged to 0. It is impossible that this number is larger for a subset fo V than for all of $\mathbb{R}^n$. However, one has to worry about the notion of "axis vector" depending on the surrounding space.

Problem 5. The KAK decomposition of $O(1, n-1)$ states that any matrix in $O(1, n-1)$ may be written as a product of three matrices, the first and last of whch is in $O(n-1)$, and the middle in a maximal diagonalizable group A. The latter has the form (for one interpretation of $O(1, n-1)$)

$$\begin{pmatrix} cosh(t) & 0 & sinh(t) \\ 0 & I & 0 \\ sinh(t) & 0 & cosh(t) \end{pmatrix}$$

where $t \in \mathbb{R}$ and I is an identity matrix.

Now, up to subsequence the K terms can be ignored since they converge to something in $O(n-1)$. Likewise, the t parameter of the A component converges up to subsequence to something in $[-\infty, \infty]$.