Representations of hyperbolic 3-manifold groups in PSL(2,R)

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"Deformation spaces of geometric structures"

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Some conjugate $g\Gamma g^{-1}$ has all matrix entries in a number field K

(K is a degree 2 extension of $\mathbb{Q}(tr\Gamma)$).

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{pmatrix}$$

$$\Gamma \hookrightarrow \mathrm{SL}(2, \mathbb{R})$$

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Which hyperbolic manifolds M^3 have trace fields with a real place?

Theorem (Calegari)

If M^3 fibers with fiber $=$ once punctured torus, then no real place.

Question

What do real place reps "look like" in $X(\pi_1(M^3), \mathsf{SL}(2, \mathbb{R}))$*?*

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For $\phi \in \mathrm{Mod}(\Sigma)$,

$$\rho \mapsto \rho \circ \phi$$

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e.g. fixed points for ϕ?

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$$t \leftrightarrow \rho(T)$$

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Question

Describe the dynamics of ϕ *near a fixed point.*

3. Transversely projective foliations

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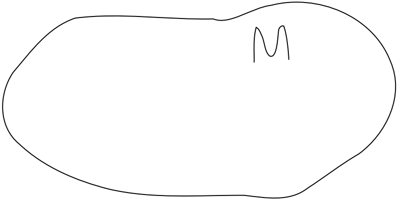
Definition

A transversely real projective codimension 1 foliation on a manifold M is



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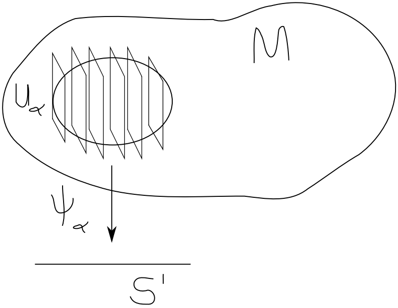

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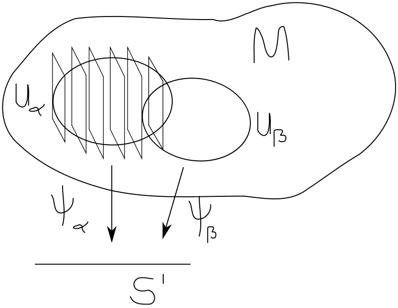

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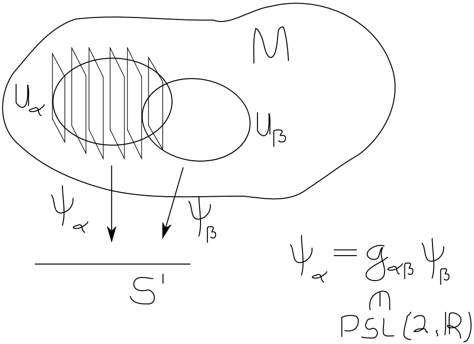

The holonomy of this foliation is a map $\pi_1(M) \to \mathrm{PSL}(2, \mathbb{R})$

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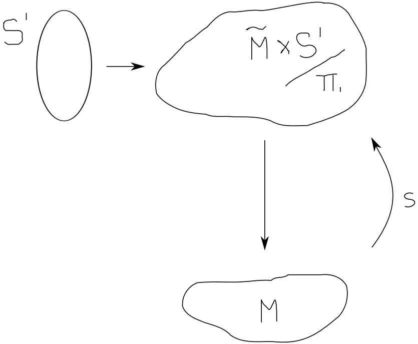

A flat S^1 bundle with holonomy in $\mathrm{PSL}(2, \mathbb{R})$, and a section s transverse to the horizontal

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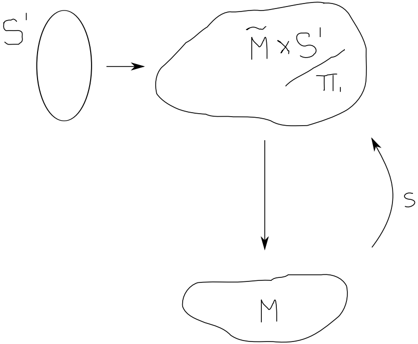

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Seifert Volume

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Which hyperbolic manifolds have non-zero Seifert volume? What is the volume associated to a given $\rho : \pi_1(M^3) \to \mathrm{PSL}(2, \mathbb{R})$*?*

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Which hyperbolic manifolds have non-zero Seifert volume? What is the volume associated to a given $\rho : \pi_1(M^3) \to \mathrm{PSL}(2, \mathbb{R})$*? Given* M^3*, what is the set of all Seifert volumes of all holonomy representations?*

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- Representations at the boundary of Dehn filling space

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Some references

1. (Arithmetic invariants, real places)

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