The hyperkähler geometry of the deformation space of complex projective structures on a surface

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Outline

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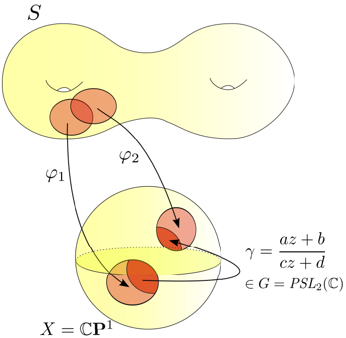

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There is a forgetful map $p : \mathcal{CP}(S) \to \mathcal{T}(S)$ where

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is the Teichmüller space of S.

Fuchsian and quasifuchsian structures

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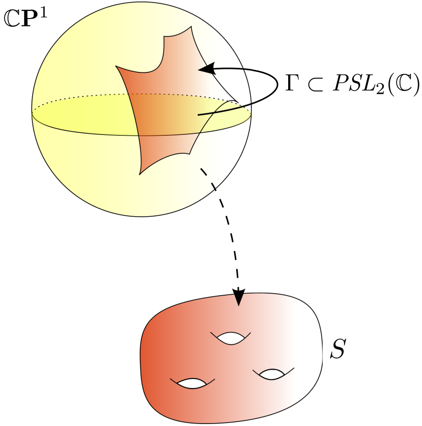

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In particular, any Riemann surface X can be equipped with a compatible $\mathbb{C}\mathbf{P}^1$-structure by the uniformization theorem:

$$X = \mathbb{H}^2$$

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Note: This defines a *Fuchsian section* $\sigma_{\mathcal{F}} : \mathcal{T}(S) \to \mathcal{CP}(S)$.

Quasifuchsian structures

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By Bers' simultaneous uniformization theorem, given two complex structures $(X^+, X^-) \in \mathcal{T}(S) \times \mathcal{T}(\overline{S})$, there exists a unique Kleinian group Γ such that:

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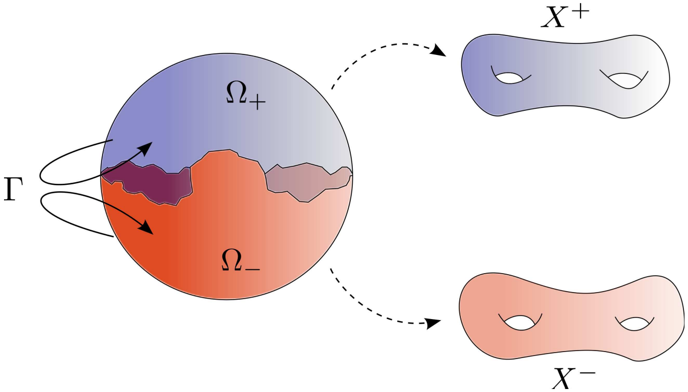

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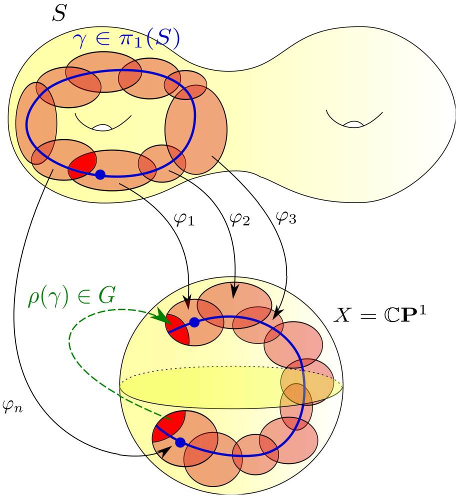

The character variety

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Holonomy defines a map

$$hol : \mathcal{CP}(S) \to \mathcal{X}(S, G)\ ;$$

where $\mathcal{X}(S, G) = \mathrm{Hom}(\pi_1(S), G)//G$ is the character variety of S.

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Abusing notations, we also let ω_G denote the complex symplectic structure on $\mathcal{CP}(S)$ obtained by pulling back ω_G by the holonomy map $hol : \mathcal{CP}(S) \to \mathcal{X}(S, G)$.

The character variety (continued)

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Theorem (Goldman)

The restriction of the complex symplectic structure on the Fuchsian slice $\mathcal{F}(S)$ is the Weil-Petersson Kähler form:

$$\sigma_{\mathcal{F}}^*(\omega_G) = \omega_{WP} \ .$$

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Theorem (Platis, L)

Complex Fenchel-Nielsen coordinates (l_i, τ_i) associated to any pants decomposition are canonical coordinates for the symplectic structure:

$$\omega_G = \sum_i dl_i \wedge d\tau_i \ .$$

Hitchin-Kobayashi correspondence

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Theorem (Hitchin, Simpson, Corlette, Donaldson)

Fix a complex structure X on S. There is a real-analytic bijection

$$H_X : \mathcal{X}^0(S, G) \xrightarrow{\sim} \mathcal{M}^0_{\mathrm{Dol}}(X, G)$$

where $\mathcal{M}^0_{\mathrm{Dol}}(X, G)$ is the moduli space of topologically trivial polystable Higgs bundles on X.

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Theorem (Hitchin)

There is a natural hyperkähler structure (g, I, J, K) on $\mathcal{M}^0_{\mathrm{Dol}}(X, G)$. The map H_X is holomorphic with respect to J. It is also a symplectomorphism for the appropriate symplectic structures.

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Theorem (Feix, Kaledin)

If M is a real-analytic Kähler manifold, then there exists a unique hyperkähler structure in a neighborhood of the zero section in T^*M such that:

- it refines the complex symplectic structure
- it extends the Kähler structure off the zero section
- the $U(1)$-action in the fibers is isometric.

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For each choice of σ, we thus get a symplectic structure ω^σ on the whole space $\mathcal{CP}(S)$ (pulling back ω_{can}) and a hyperkähler structure on some neighborhood of the Fuchsian slice.

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$\mathcal{CP}(S) \approx^{\sigma} T^*\mathcal{T}(S)$ is a complex symplectomorphism iff $d(\sigma - \sigma_{\mathcal{F}}) = \omega_{WP}$ (on $\mathcal{T}(S)$).

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  - This is similar to the situation we saw with the Hitchin-Kobayashi correspondence.
    Quiz : what is a significant difference though?

The minimal surface parametrization

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The space of almost-Fuchsian structures $\mathcal{AF}(S) \subset \mathcal{QF}(S)$ is a neighborhood of the Fuchsian slice such that if $Z \in \mathcal{AF}(S)$, the hyperbolic 3-manifold associated to Z contains a unique minimal surface Σ.

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Again, one can use this "minimal surface parametrization" to pull back the hyperkähler structure of $T^*\mathcal{T}(S)$ on $\mathcal{CP}(S)$.

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Theorem (L)

The minimal surface parametrization $\mathcal{AF}(S) \xrightarrow{\sim} T^*\mathcal{T}(S)$ is a real symplectomorphism (for the appropriate symplectic structures).