DARREN LONG MINI-COURSE SOLUTIONS

#1

Let G be the fundamental group of a Hausdorff, locally compact topological space M_G, then G is residually finite if and only if for each compact set C in in the universal cover, $\tilde{M}_G$, there exists a finite sheeted cover, M_C, of M_G such that C embeds in M_C.

Proof. Assume that M_G has the geometric mapping property mentioned above, and let γ be a loop in M_G representing $1 \neq g \in G$. Since g is non trivial it has a lift that is a path in $\tilde{M}_G$, call it $\tilde{g}$. $\tilde{g}$ is compact when thought of as a subset of $\tilde{M}_G$, and so we can find a finite sheeted cover M_g, where $\tilde{g}$ embedds (i.e. it is still a path and not a loop). By elementary covering theory, we see that this means that g is not contained in the finite index subgroup of G that corresponds to the cover M_g, and so g is residually finite.

Conversely, suppose that G is residually finite. We claim that if $S = \{g_1, \ldots, g_n\}$ is a finite list of nontrivial elements in G, then there exists a homomorphism $\phi : G \to A$, where A is finite and $1 \notin \phi(S)$. For each g_i, residual finiteness tells us that there exists $\phi_i : G \to A_i$, where A_i is finite and $\phi_i(g_i) \neq 1$. We can take ϕ to be the quotient homomorphism from G to $G/(\mathrm{Ker}\phi_1 \cap \ldots \cap \mathrm{Ker}\phi_\mathrm{n})$, thus proving the claim.

Next, let C be a compact subset of $\tilde{M}_G$. Since G acts properly discontinuously on $\tilde{M}_G$ by deck transformations we see that the set $S = \{g \in G \mid gC \cap C \neq \emptyset\}$ is finite. Applying the claim we see that we can find a finite sheeted cover, M_S, whose fundamental group does not contain the elements of S. Since M_S is the quotient of $\tilde{M}_G$ by $\pi_1(M_S)$ we see that the obstruction to C embedding in M_S is precisely elements $h \in \pi_1(M_S)$ such that $hC \cap C \neq \emptyset$, and so by our choice of M_S we see that C embeds. □

#2

Let H be a finite index subgroup of G, show that if H is subgroup separable then so is G.

Proof. We will prove the proposition using the language of the profinite topology on G and H. Recall that the profinite topology on G is given by declaring that finite index subgroups of G are the neighborhoods of the identity in G. I claim that a subgroup, K is separable in G is the same as being closed in this topology. To see this observe that if K is separable and $g \notin K$ then there exists a finite index subgroup, K', of G that contains K but not g. Thus the coset gK' is a neighborhood of g. Since $g \notin K'$ we see that gK' is disjoint from K' and hence disjoint from K. What we have just shown is that the complement of K in G is open, thus proving the claim. Reversing the steps of the previous argument shows that if K is closed in the profinite topology then it is separable.

Next, I claim that the subspace topology on H induced by its inclusion in G is equivalent to the profinite topology on H. To see this observe that if K is an open neighborhood in the subspace topology on H then there exists an open set K' in G such that $K = H \cap K'$, (technically K' is a union of neighborhoods, but for simplicity assume that K' is a single neighborhood of the identity). Since K' and H both have finite index in G their intersection also has finite index in H, and so we see that $H \cap K'$ is open in the profinite topology on H. On the other hand if K is an open neighborhood of the identity in the profinite topology on H then K has finite index in H. Without loss of generality we can assume that K is normal in G. Since H has finite index in G and K is normal in G we see that KH is a finite index subgroup of G, and is thus open in the profinite topology on G. Finally, observe that $KH \cap H = H$ and so H is open in the subspace topology on H, thus proving the claim.

Before proving that G is subgroup separable we make the simplifying assumption that H is normal in G. This is fine because subgroups of H are also subgroup separable and the normal core of H is a finite index normal subgroup of G contained in H. Let K be a finitely generated subgroup of G. H has finite index in the subgroup KH and so by the second isomorphism theorem $K \cap H$ has finite index in K. Since finite index subgroups of finitely generated groups are also finitely generated we see that $K \cap H$ is also finitely generated. By our assumption on H we see that $K \cap H$ is closed in the profinite topology on H and hence closed in the subspace topology on H, but this implies that K is closed in the profinite topology on G, and so K is separable in G. □

5

Let G be a residually finite group and A a maximal abelian subgroup of G. Show that A is separable in G. Show that for a finite volume hyperbolic 3-manifold M cyclic subgroups generated by a primitive hyperbolic element and cusp subgroups are both separable.

Proof. If G is abelian then the result is trivial, so assume that G is non-abelian. We begin with a useful lemma that Darren mentioned in his course.

Lemma 0.1. *Let G be a residually finite group and f an automorphism of G, then the fixed point set of f is separable in G.*

Proof. Let H be the fixed point set of f, and let $g \notin H$. By residual finiteness we know that there exists $\phi : G \to A$, where A is finite and $\phi(gf(g^{-1})) \neq 1$. Next, define $\psi : G \to A \times A$ by $g \mapsto (\phi(g), \phi(f(g)))$. Since H is fixed by f we see that ψ maps H into the diagonal of $A \times A$. If we can show that ψ does not map g into the diagonal then we will have separated g from H in a finite quotient. Assume for contradiction that $\psi(g)$ also maps into the diagonal. In this case we have $\psi(g) = (\phi(g), \phi(f(g))) = (\phi(g), \phi(g))$, which implies that $\phi(g) = \phi(f(g))$, however by our choice of ϕ this is a contradiction. □

We will now use the lemma to prove separability of A. Let $g \notin A$. Since A is maximal abelian we can select $a \in A$ such that $ga \neq ag$. Let f_a be the map given by $h \mapsto aha^{-1}$. The fixed set of f_a is the centralizer, $C_G(a)$, of a in G. Two important facts about $C_G(a)$ are that it contains A and that it does not contain g. By the lemma we see that $C_G(a)$ is separable in G, and so there is some finite index subgroup of G that separates g from $C_G(a)$, however since $A \leq C_G(a)$ this same subgroup also separates g from A, and so A is also separable.

In a finite volume hyperbolic 3-manifold group both cyclic subgroups generated by primitive hyperbolic elements and cusp subgroups are maximal abelian, and by the previous argument they are separable.

□

#6

Let G be the fundamental group of a finite volume hyperbolic 3-manifold, and H be a maximal totally geodesics subgroup. Show that H is separable in G.

Proof. Since H is totally geodesic, there exists $a \in PSL_2(\mathbb{C})$ such that $aHa^{-1} \leq PSL_2(\mathbb{R})$. Suppose that $g \notin H$, then by maximality of H, $g - \overline{g} \neq 0$, and so there is some non-zero entry of $g - \overline{g}$, call it x. Let $\mathcal{R}$ be the subring of $\mathbb{C}$ generated by x the entries of the generators of G and their conjugates. The ring $\mathcal{R}$ has the following two properties: first,for any non-zero element, y, there is a maximal ideal of $\mathcal{R}$ not containing y. Second, for any maximal ideal $\mathcal{M}$, $\mathcal{R}/\mathcal{M}$ is a finite field. Let $\mathcal{M}$ be a maximal ideal of $\mathcal{R}$ that does not contain x and let $\phi : G \to PSL_2(\mathcal{R}/\mathcal{M}) \times PSL_2(\mathcal{R}/\mathcal{M})$ be given by $g \mapsto (\tilde{g}, \tilde{\overline{g}})$, (where $\tilde{g}$ is the matrix obtained by reducing the coefficients of g modulo $\mathcal{M}$). Under ϕ, H is sent to the diagonal, but by our choice of $\mathcal{M}$ we see that g is mapped away from the diagonal, and so we have separated g from H in a finite quotient. □