EXERCICES – MAXIMAL REPRESENTATIONS

1. Finitely generated subgroups of surface groups

The aim of these exercices is to prove Scott's Theorem: every finitely generated subgroup of a surface group is *almost geometric*.

A subgroup of a surface group $\pi_1\Sigma$ is said *geometric* if it is of the form $\pi_1 Y$ for an incompressible surface Y of Σ (i.e. $\pi_1 Y$ injects in $\pi_1\Sigma$). A subgroup S of a surface group $\pi_1\Sigma$ is called *almost geometric* if there is a finite covering $\Sigma' \to \Sigma$ and an incompressible surface Y of Σ' such that $S = \pi_1 Y < \pi_1\Sigma' < \pi_1\Sigma$.

The main idea is to relate this result for a surface group G to an algebraic property of G: G is locally extended residually finite.

Recall that a group G is *residually finite* (RF) if for any non-trivial element g of G, there is a finite index subgroup $G' < G$ such that $g \notin G'$. When S is subgroup of G, G is called *S-residually finite* (S-RF) if for any $g \in G \setminus S$, there is a finite index subgroup G' containing S but not g. The group G is called *extended residually finite* (ERF) if it is S-RF for any subgroup S of G. It is called *locally extended residually finite* (LERF), if it is S-RF for any finitely generated subgroup S.

1. Recast those properties (RF, ERF, LERF) in term of the profinite topology of G (find a friend to explain to you the profinite topology).

2. If G is RF, ERF, or LERF, proove that any subgroup of G has the same property. If K is a group containing G as a finite index subgroup, proove that K has the same property.

3. Let X be a Hausdorff topological space with regular covering $\tilde{X}$ with covering group G. Let C be a compact subset of $\tilde{X}$. Proove that $\{g \in G \mid gC \cap C \neq \emptyset\}$ is finite.

4. Let X, $\tilde{X}$ and G as above. Proove that the following conditions are equivalent:

(1) G is RF.
(2) For any compact subset C of $\tilde{X}$, there is a finite index subgroup G' of G such that $gC \cap C = \emptyset$ for any non-trivial element g of G'.
(3) For any compact subset C of $\tilde{X}$, the projection $\tilde{X} \to X$ factors through a finite covering X' such that C projects by a homeomorphism into X'.

5. Let X, $\tilde{X}$ and G as above. Proove that G is LERF if and only if, given any finitely generated subgroup S of G and a compact subset C of $\tilde{X}/S$, there is a finite covering X' of X such that the projection $\tilde{X}/S \to X$ factors through X' and such that C projects by a homeomorphism into X'.

6. Let Σ be a connected surface such that $\pi_1 F$ is finitely generated and let C be a compact subset of F. Proove that there is a compact, connected, incompressible subsurface Y of F containing C and such that $\pi_1 Y \to \pi_1 F$ is an isomorphism.

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7. Let F be a surface. Proove that $\pi_1 F$ is LERF if and only if, given any finitely generated subgroup S of $\pi_1 F$ and g in $\pi_1 F \setminus S$ there is a finite covering F' of F such that $\pi_1 F'$ contains S and not g and such that S is geometric in F'.

Let $\mathbb{H}$ be the hyperbolic plane. Recall that there exists P a right angled regular pentagon in $\mathbb{H}$. Let G be the group generated by the reflections through the sides of P. Let $\mathcal{L}$ the geodesic lines appearing in the tesselation $\mathbb{H} = G \cdot P$ and let $\mathcal{H}$ be the set of half-spaces whose boundary is a line L of $\mathcal{L}$.

8. Proove that every surface group $\pi_1\Sigma_g$ ($g \geq 2$) is a finite index subgroup of G.

Define R the following constant: x is point in $\mathbb{H}$ and r_1 and r_2 two geodesic rays staring at x and orthogonal at x and l is the geodesic asympotic to both r_1 and r_2, then R is the distance from x to l.

9. Let S be a finitely generated subgroup of G.
Let $CH(S)$ be the convex hull of S and let $\mathcal{CH}(S)$ be the intersection of the half-planes in $\mathcal{H}$ containing $CH(S)$. Proove that $\mathcal{CH}(S)$ is contained is the R-neighborhood of $CH(S)$.

10. Proove that $\mathcal{CH}(S)/S$ is compact.

11. Let Ω be the (infinitely generated) reflection groups through the sides of $\mathcal{CH}(S)$ and let G' be the group generated by Ω and S.
Proove that Ω is normalized by S, that $G'/\Omega \simeq S$ and that G' is of finite index in G.

12. Prove that G' is RF.

13. Denote by r the projection $G' \to G'/\Omega = S$ so that $r|_S = \mathrm{id}$. Let $f : G' \to G' \mid g \mapsto g^{-1} \cdot r(g)$. Prove that f is continuous in the profinite topology, that $S = f^{-1}(\{1\})$ and that S is closed in the profinite topology. Conclude.

2. Topology of $\mathrm{Homeo}^+(S^1)$

Let $S^1 = \mathbf{R}/\mathbf{Z}$ be the circle and let $G = \mathrm{Homeo}^+(S^1)$ be the group of orientation preserving homeomorphisms of S^1. Let $\tilde{G}$ be the group of homeomorphisms of $\mathbf{R}$ commuting with integer translations, i.e. homeomorphisms f such that $f(x+1) = f(x) + 1$ for any x in $\mathbf{R}$.

14. Define the topology on G and $\tilde{G}$.

15. Exhibit a "natural" map from $\tilde{G}$ to G. What is the kernel of this homomorphism? Proove that it is surjective.

16. Proove that f is in $\tilde{G}$ if and only if $f : \mathbf{R} \to \mathbf{R}$ is increasing and commutes with $x \mapsto x + 1$.

17. Proove that $\tilde{G}$ is contractible (hint: convex combinations). Conclude that $\tilde{G} \to G$ is the universal covering of G.

18. Proove that the stabilizer in $\tilde{G}$ of a point in $\mathbf{R}$ is contractible. Proove that the stabilizer in G of a point in S^1 is contractible. Determine the homotopy type of G.

19. Define the (sub)groups $\mathrm{Diffeo}^+_k(S^1)$ ($k \in \mathbf{N}^*$), $\mathrm{Diffeo}^+_\infty(S^1)$, $\mathrm{Diffeo}^+_\omega(S^1)$ and their universal coverings. What are the homotopy types of those groups? What are their topologies?

20. Prove that $\mathrm{PSL}(2,\mathbf{R})$ acts faithfully on S^1, i.e. that there is an injective homomorphism $\mathrm{PSL}(2,\mathbf{R}) \to \mathrm{Diffeo}^+_\omega(S^1)$.

21. What is $\pi_1\mathrm{PSL}(2,\mathbf{R})$? Let $\mathrm{PSL}(2,\mathbf{R})_{(k)}$ be the unique (proove uniqueness!) connected k-covering of $\mathrm{PSL}(2,\mathbf{R})$ ($k \in \mathbf{N}^*$). Which classical group is $\mathrm{PSL}(2,\mathbf{R})_{(2)}$? Proove that there exists $\mathrm{PSL}(2,\mathbf{R})_{(k)} \hookrightarrow \mathrm{Diffeo}^+_\omega(S^1)$. Why are all those subgroups of $\mathrm{Homeo}^+(S^1)$ not conjugated?

22. Determine the topology of $\mathrm{PSL}2,\mathbf{R}$ and of $\widetilde{\mathrm{PSL}2,\mathbf{R}}$.

23. Proove the lemmata about $\underline{m}$ and $\overline{m}$ stated in the first lectures.

3. Topology of $\mathrm{Sp}(2n,\mathbf{R})$

Let ω be the symplectic form on $\mathbf{R}^{2n}$ whose matrix in the canonical basis is $\left(\begin{smallmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{smallmatrix}\right)$. Let $\mathrm{Sp}(2n,\mathbf{R})$ be the group of ω-symplectic linear transformations of $\mathbf{R}^{2n}$ and let $K = \mathrm{Sp}(2n,\mathbf{R}) \cap \mathrm{O}(2n)$.

24. What are the Lie algebras $\mathfrak{sp}(2n,\mathbf{R})$ and $\mathfrak{k}$ of $\mathrm{Sp}(2n,\mathbf{R})$ and K? (as subalgebras of $\mathfrak{gl}(2n,\mathbf{R}) = M_{2n}(\mathbf{R})$).

25. Proove that $b(A,B) = \mathbf{tr}(AB)$ is an $\mathrm{Sp}(2n,\mathbf{R})$-invariant non-degenerate quadratic form on $\mathfrak{sp}(2n,\mathbf{R})$ and on $\mathfrak{k}$. Determine $\mathfrak{p} = \mathfrak{k}^{\perp b}$. What is the signature of $b|_\mathfrak{k}$? of $b|_\mathfrak{p}$?

26. Proove that $K \times \mathfrak{p} \to \mathrm{Sp}(2n,\mathbf{R}) \mid (k,X) \mapsto k\exp(X)$ is a diffeomorphism.

27. Proove that $K \simeq \mathrm{U}(n)$. What is $\pi_1 K$? What is $\pi_1\mathrm{Sp}(2n,\mathbf{R})$?

A subspace V of $\mathbf{R}^{2n}$ is called *isotropic* or ω-isotropic if $\omega|_V = 0$.

28. What is the maximal dimension of an isotropic subspace?

A subspace is called *Lagrangian* if it is isotropic and of maximal dimension. Let $\mathcal{L}$ be the set of all Lagrangian subspaces.

29. Prove that $\mathcal{L}$ is homogenous under the action of $\mathrm{Sp}(2n,\mathbf{R})$. Define the (a?) topology on $\mathcal{L}$. Prove that the $\mathrm{Sp}(2n,\mathbf{R})$-action is continuous.

Let $\mathcal{L}^+$ be the set of *oriented* Lagrangians.

30. Prove that $\mathcal{L}^+$ is $\mathrm{Sp}(2n,\mathbf{R})$-homogenous. What is $\pi_1\mathcal{L}^+$? Prove that any orbital application $\mathrm{Sp}(2n,\mathbf{R}) \to \mathcal{L}^+$ induces an isomorphism $\pi_1\mathrm{Sp}(2n,\mathbf{R}) \to \pi_1\mathcal{L}^+$.

4. Maximal Representations

31. Let $\pi_1\Sigma_g$ be a surface group and $\rho : \pi_1\Sigma_g \to \mathrm{SL}(2,\mathbf{R})$ be a maximal representation and let τ be the homomorphism $\mathrm{SL}(2,\mathbf{R}) \to \mathrm{Sp}(2n,\mathbf{R})$ sending $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} a\mathrm{I}_n & b\mathrm{I}_n \\ c\mathrm{I}_n & d\mathrm{I}_n \end{smallmatrix}\right)$. Prove that $\tau \circ \rho$ is maximal.

32. Let $\tau : \mathrm{SL}(2,\mathbf{R}) \to \mathrm{GL}(2n,\mathbf{R})$ be the (unique!) irreducible homomorphism. Prove that (up to conjugation) $\tau(\mathrm{SL}(2,\mathbf{R})) \subset \mathrm{Sp}(2n,\mathbf{R})$. Prove that $\tau \circ \rho$ is maximal.

33. Let n be an integer and choose $n = n_1 + \cdots + n_k$ a decomposition of n, i.e. each n_i is an integer. Let $H = \mathrm{Sp}(2n_1, \mathbf{R}) \times \cdots \times \mathrm{Sp}(2n_k, \mathbf{R})$ as a subgroup of $\mathrm{Sp}(2n, \mathbf{R})$. Let $\rho_1, \ldots, \rho_k$ be k representations of $\pi_1\Sigma_g$, $\rho_i : \pi_1\Sigma_g \to \mathrm{Sp}(2n_i, \mathbf{R})$ and let $\rho = (\rho_1, \ldots, \rho_k) : \pi_1\Sigma_g \to H < \mathrm{Sp}(2n, \mathbf{R})$ be the corresponding representation. Proove that ρ is maximal if and only if, for all i, ρ_i is maximal.

34. Determine all the possible morphisms $\tau : \mathrm{SL}(2, \mathbf{R}) \to \mathrm{Sp}(2n, \mathbf{R})$ (up to conjugation). Let $\rho : \pi_1\Sigma_g \to \mathrm{SL}(2, \mathbf{R})$ be a maximal representation. For which τ is $\tau \circ \rho$ maximal?

35. Let $\mathrm{Sp}(2n, \mathbf{R})_{(k)}$ be the connected k-covering of $\mathrm{Sp}(2n, \mathbf{R})$.

Proove that $\pi_1\mathrm{Sp}(2n, \mathbf{R})_{(k)} = \mathbf{Z}$ and that the map $\mathbf{Z} = \pi_1\mathrm{Sp}(2n, \mathbf{R})_{(k)} \to \pi_1\mathrm{Sp}(2n, \mathbf{R}) = \mathbf{Z}$ is the multiplication by k.

36. Let $\rho : \pi_1\Sigma_g \to \mathrm{Sp}(2n, \mathbf{R})_{(k)}$ be a homomorphism. Define $T(\rho) \in \pi_1\mathrm{Sp}(2n, \mathbf{R})_{(k)} = \mathbf{Z}$ its invariant. Prove that $|T(\rho)| \leq n(g-1)/k$. Proove that there are maximal representations $\pi_1\Sigma_g \to \mathrm{Sp}(2n, \mathbf{R})_{(k)}$ if and only if k divides $n(g-1)$.

The aim of the remaining exercices of this section is to give a hint of the existence of maximal representations for surfaces with boundaries.

37. Let γ be a separating simple closed curve of a surface Σ of genus g. Hence Σ is the union $\Sigma_l \cup_\gamma \Sigma_r$ and, by the Van Kampen Theorem, $\pi_1\Sigma$ is isomorphic to $\pi_1\Sigma_l \star_{\pi_1\gamma} \pi_1\Sigma_r$. Let $\rho : \pi_1\Sigma \to \mathrm{Sp}(2n, \mathbf{R})$ be a representation. Since $\pi_1\Sigma_l$ and $\pi_1\Sigma_r$ are gree groups, $\rho|_{\pi_1\Sigma_l}$ and $\rho|_{\pi_1\Sigma_r}$ lift to representations $\tilde\rho_l : \pi_1\Sigma_l \to \widetilde{\mathrm{Sp}(2n, \mathbf{R})}$ and $\tilde\rho_r : \pi_1\Sigma_r \to \widetilde{\mathrm{Sp}(2n, \mathbf{R})}$.

Let $z \in \pi_1\mathrm{Sp}(2n, \mathbf{R}) < \widetilde{\mathrm{Sp}(2n, \mathbf{R})}$ be the generator.

Proove that $\tilde\rho_l(\gamma)$ and $\tilde\rho_r(\gamma)$ are independant of the chosen lifts. Proove that $\tilde\rho_l(\gamma) = z^{T(\rho)}\tilde\rho_r(\gamma)$ where $T(\rho) \in \mathbf{Z}$ is the Toledo number of ρ.

38. Let $r : \mathrm{Sp}(2n, \mathbf{R}) \to \mathbf{R}/\mathbf{Z} = \mathrm{U}(1)$ be the map obtained by composing the projection to the first factor $\mathrm{Sp}(2n, \mathbf{R}) \simeq \mathrm{U}(n) \times \mathfrak{p} \to \mathrm{U}(n)$ with the determinant $\det : \mathrm{U}(n) \to \mathrm{U}(1)$. Let $\tilde r : \widetilde{\mathrm{Sp}(2n, \mathbf{R})} \to \mathbf{R}$ be the lift of r.

Proove that r induces an isomorphism at the level of fundamental groups. What is $\tilde r(z)$?

Proove that, for all a, b in $\widetilde{\mathrm{Sp}(2n, \mathbf{R})}$, $|\tilde r(ab) - \tilde r(a) - \tilde r(b)| \leq n/2$ (enough is to find a bound).

Let $\tilde r^h : \widetilde{\mathrm{Sp}(2n, \mathbf{R})} \to \mathbf{R}$ defined by $\tilde r^h(a) = \lim \tilde r(a^k)/k$. Proove that $\tilde r^h$ is well defined and continuous. Proove that $|\tilde r^h(ab) - \tilde r^h(a) - \tilde r^h(b)|$ is uniformly bounded and that $\tilde r^h(a^k) = k\tilde r^h(a)$. Proove that $\tilde r^h(az) = \tilde r^h(a) + 1$.

39. Let $\Sigma, \gamma, \Sigma_l, \Sigma_r$ be as above. Let $\rho_l : \pi_1\Sigma_l \to \mathrm{Sp}(2n, \mathbf{R})$ be a representation and choose a lift $\tilde\rho_l : \pi_1\Sigma_l \to \widetilde{\mathrm{Sp}(2n, \mathbf{R})}$.

Proove that $\tilde\rho_l(\gamma)$ is independant of the choice of $\tilde\rho_l$ and hence that $T(\rho_l) := \tilde r^h(\tilde\rho_l(\gamma))$ is well defined.

Proove that $|T(\rho_l)| \leq n(g_l - 1/2)$ (use next exercise). We say that ρ_l is maximal if and only if $T(\rho_l) = n(g_l - 1/2)$.

40. Let $\rho : \pi_1\Sigma \to \mathrm{Sp}(2n, \mathbf{R})$ be a representation and let $\rho_l = \rho|_{\pi_1\Sigma_l}$ and $\rho_r = \rho|_{\pi_1\Sigma_r}$. Proove that $T(\rho) = T(\rho_l) + T(\rho_r)$. Proove that ρ is maximal if and only if ρ_l and ρ_r are maximal.

41. Construct maximal representations with the help of the preceeding exercices.

5. Cohomology of Groups, Continuous Cohomology, Bounded Cohomology

These sets of exercises develop the necessary material for defining cohomology, continuous cohomology and bounded cohomolgy of groups and topological groups through the use of resolutions.

Cohomology of Groups

Let G be a group. A G-module is said to be *injective* if for any injective G-morphism $\iota : A \to B$ and any G-morphism $\alpha : A \to U$, there exists a G-morphism $\beta : B \to U$ extending α: $\beta \circ \iota = \alpha$. An *injective resolution* of a G-module V is an exact sequence of G-morphisms:

$$0 \to V \xrightarrow{\epsilon} A^0 \xrightarrow{d^1} A^1 \xrightarrow{d^2} \cdots \to A^{q-1} \xrightarrow{d^q} A^q \to \cdots$$

where all A^q are injective. It is called a *resolution* without this condition on the A^q, and if furthermore the sequence is not anymore exact but has simply $d^{q+1} \circ d^q = 0$, it is now called a *differentiable complex*.

The cohomology $H^\bullet(A^\bullet)$ of a differentiable complex $(A^\bullet, d^{\bullet+1})$ is the sequences of groups $H^i(A^\bullet) = \ker d^{i+1}/\mathrm{im}(d^i)$ (G is not involved in this definition).

A *morphism* (or G-morphism) between two differentiable complexes $A^\bullet$ and $B^\bullet$ is a sequence of G-morphisms $\alpha^q : A^q \to B^q$ such that $d^q \circ \alpha^{q-1} = \alpha^q \circ d^q$.

An *homotopy* between two G-morphisms $\alpha^\bullet$ and $\beta^\bullet$ is a sequence of G-morphisms $h^q : A^q \to B^{q-1}$ such that $\alpha^q - \beta^q = d^q \circ h^q + h^{q+1} \circ d^{q+1}$.

42. Let $A^\bullet$ and $B^\bullet$ be two differentiable complexes with $A^{-1} = B^{-1} = V$, $d^0 : V \to A^0$ and $d^0 : V \to B^0$ injective and with $A^q = B^q = 0$ for all $q < -1$. Suppose that $A^\bullet$ is exact (is it really necessary?) and that, for all $q \geq 0$, B^q is injective. Proove that there exists a G-morphism $\alpha^\bullet : A^\bullet \to B^\bullet$ with $\alpha^{-1} = \mathrm{id}_V : A^{-1} \to B^{-1}$. Proove that any two such G-morphisms are homotopic.

43. Proove that any two injective resolutions $A^\bullet$ and $B^\bullet$ of a G-module V are homotopy equivalent, i.e. that there exist $\alpha^\bullet : A^\bullet \to B^\bullet$ and $\beta^\bullet : B^\bullet \to A^\bullet$ such that $\beta^\bullet \circ \alpha^\bullet$ is homotopic to $\mathrm{id}_{A^\bullet}$ and $\alpha^\bullet \circ \beta^\bullet$ is homotopic to $\mathrm{id}_{B^\bullet}$.

44. Proove that for any two injective resolutions of V, the cohomology of the complexes of G-invariants elements, $(A^{\bullet G})_{\bullet \geq 0}$ and $(B^{\bullet G})_{\bullet \geq 0}$, are canonically isomorphic: $H^\bullet(A^{\bullet G}) \simeq H^\bullet(B^{\bullet G})$.

45. Let $A^q = \mathcal{F}(G^{q+1}, V)$ be the G-module of functions from G^{q+1} to a G-module V. The G-action on A^q is defined so that the following calculation rule is valid: $g \cdot (f(g_0, \ldots, g_q)) = (g \cdot f)(gg_0, \ldots, gg_q)$ for any g in G, any f in A^q and any $(g_0, \ldots, g_q)$ in G^{q+1}.

Proove that A^q is an injective G-module.

Let $\epsilon : V \to A^0$ be the map sending v to the function constant equal to v. And let $d^q : A^{q-1} \to A^q$ defined by:

$$d^q f(g_0, \ldots, g_q) = \sum_{i=0}^{q} (-1)^i f(g_0, \ldots, \hat{g}_i, \ldots, g_q).$$

Proove that $A^\bullet$ is a resolution of V, it is called the *homogenous cochains* resolution.

What is the group cohomology of G with coefficient in V?

Continuous Cohomology

Let G be a locally compact second countable topological group.

Let $\mathcal{C}_G$ be the category of locally convex, locally complete (meaning?) Hausdorff topological vector space V with a continuous G-action, i.e. $G \times V \to V$ is continuous.

A G-morphism (in the category $\mathcal{C}_G$, i.e. continuous!) $f : A \to B$ is called *strong* if $\ker(f)$ and $\operatorname{im}(f)$ are closed topological direct summands and if f induces an isomorphism between $A/\ker(f)$ and $f(A)$.

There is a corresponding notion of strong sequences and strong exact sequences.

An element U in $\mathcal{C}_G$ is called *s-injective* if for any $\iota : A \to B$ strong injective and any $\alpha : A \to U$ there exists an extension $\beta : B \to U$ ($\beta \circ \iota = \alpha$). ($\iota$, α and β are G-morphisms).

46. Define the notion of continuously s-injective resolution.

47. Construct the continuous cohomology of G.

48. Define the complex of continuous homogenous cochains. Show it is a continuously s-injective resolution.

Bounded Cohomology

Let G be a locally compact second countable topological group. One want to work with Banach spaces here, however imposing continuity of the G-action would be too restrictive. In this section, we work with Banach spaces V with a homomorphism $G \to \operatorname{Aut}(V)$, V is called continuous when $G \times V \to V$ is continuous.

A G-morphism $f : A \to B$ is called *admissible* if there is a continuous linear map $\alpha : B \to A$ (not a G-morphism!) such that $||\alpha|| \leq 1$ and $f\alpha f = f$.

A G-Banach space U is called *relatively injective* if for any injective admissible G-morphism $\iota : A \to B$ between *continuous* G-Banach spaces and any $\alpha : A \to U$, there is $\beta : B \to U$ such that $\beta \circ iota = \alpha$ and $||\beta|| \leq ||\alpha||$.

49. For a G-Banach space E, show that there exists a maximal subspace $\mathcal{C}E$ where the G-action is continuous, proove that $e \in \mathcal{C}E \Leftrightarrow$ the orbital map $G \to E \mid g \mapsto g \cdot e$ is continuous. Show that $E^G \subset \mathcal{C}E$.

A differentiable complex $A^\bullet$ of Banach space has a *contracting homotopy* if there is an homotopy $h^\bullet$ between $\mathrm{id}_{A^\bullet}$ and $0_{A^\bullet}$ such that, for all q, $||h^q|| \leq 1$.

A resolution $A^\bullet$ is called *strong* if $\mathcal{C}A^\bullet$ has a contracting homotopy.

50. Define relatively injective strong resolutions of a Banach G-module V.

51. Define the bounded cohomology with coefficient in V.

52. Show that the complex of bounded continuous homogenous cochains (with the sup norm) is such a resolution.