# A Non-Injective Skinning Map with a Critical Point

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He re-phrased the "glueing problem" in terms of finding a fixed point of a certain holomorphic map on a Teichmüller space.

Given a geometrically finite hyperbolic 3-manifold, $M$, with incompressible boundary $\Sigma$, the skinning map $\sigma_M$ is a holomorphic map

$$\sigma_M : \mathcal{T}(\Sigma) \to \mathcal{T}(\bar{\Sigma})$$

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**Theorem (Dumas, 2011):**

$\sigma_M$ is open and finite-to-one.

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We present a negative answer to the questions above:

**Theorem (G.):**

There exists a hyperbolic structure on a genus-2 handlebody, with two rank-1 cusps, whose skinning map is non-injective and has a critical point.

# Outline

Background

- Invariants of Quasi-Fuchsian and Geometrically Finite groups
- Pared 3-manifolds

The skinning map $\sigma_M$ and a useful lemma

- The definition
- A Symmetry Lemma

The Example

- Glueing an Octahedron
- 4-Punctured Spheres
- The Path of Quasi-Fuchsian Groups
- Non-monotonicity
- Further Questions

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Every hyperbolic 3-manifold is determined by the conjugacy class of a Kleinian group, so we may blur the distinction between $\Gamma$ and $\mathbb{H}^3/\Gamma$.

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- The bending laminations $\lambda_\Gamma$ on the convex core boundary surfaces (when non-empty)
- The conformal boundary surfaces (when non-empty), quotients of components of $\Omega_\Gamma$ by $\Gamma$

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Notation:

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If $\Omega_0$ is a component of $\Omega_{\hat{\Gamma}}$, $\hat{\Gamma}$ a geometrically finite Kleinian group, then $\mathrm{Stab}(\Omega_0)$ is quasi-Fuchsian.

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This Important Fact (IF) will make a couple of appearances. The proof uses 3-manifold topology.

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The geomeric invariants of a quasi-Fuchsian group can be grouped into a convenient cartoon.

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**Theorem (Ahlfors, Bers, Marden, Sullivan)**

$\mathcal{GF}(M) \cong \prod_i \mathcal{T}(\Sigma_i)$

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- The IF: $\mathrm{Stab}(\Omega_i) < \hat{\Gamma}$ is quasi-Fuchsian

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$r_i$ lands in $\mathcal{QF}(\Sigma_i)$ because of the IF.

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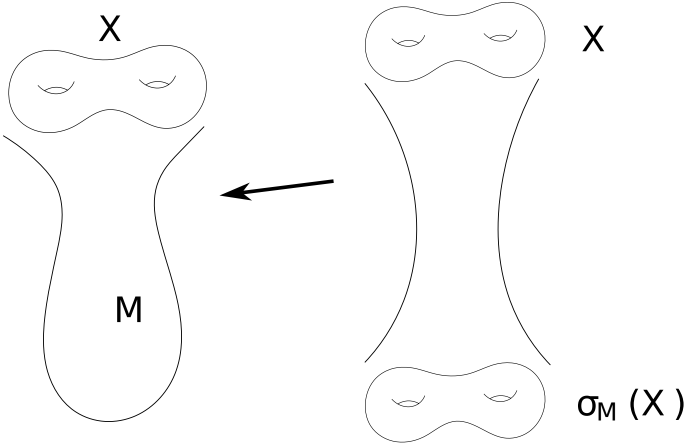

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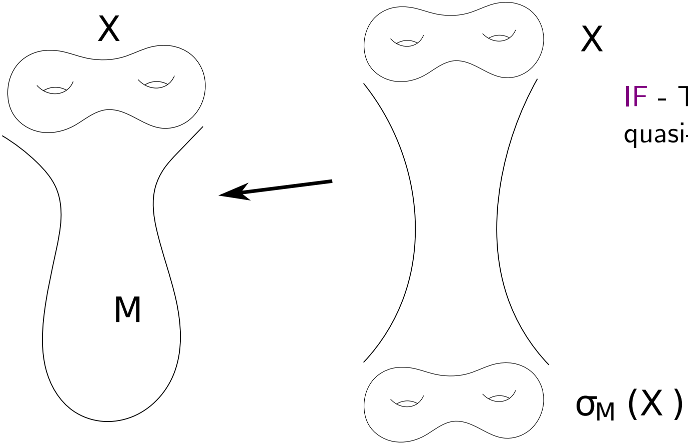


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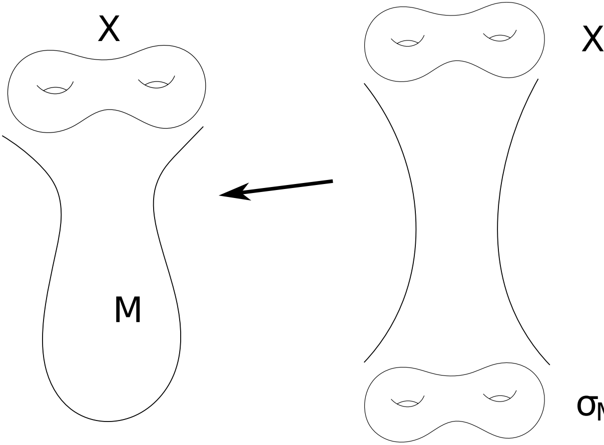


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Note that $\sigma_M$ depends only on the topology of $(M, \Sigma)$.

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Suppose $M$ is quasi-Fuchsian. That is, $M \cong \mathbb{H}^3/\Gamma$, for $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ quasi-Fuchsian. Topologically, $M \cong \Sigma \times [0, 1]$.

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In this case,

$$\sigma_M(X, \bar{Y}) = (\bar{Y}, X)$$

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This lemma is an immediate consequence of the observation that $\sigma_M$ is $\mathrm{MCG}^*(M)$-equivariant.

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Non-monotonicity restricted to a real one-dimensional submanifold guarantees the existence of a critical point.

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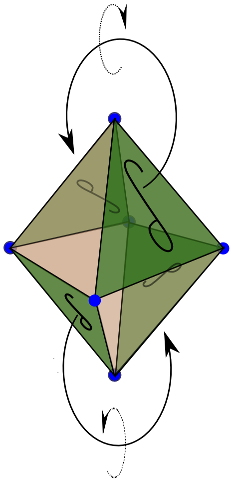

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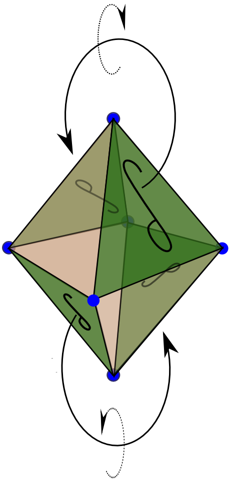

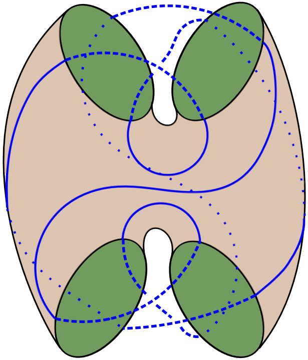

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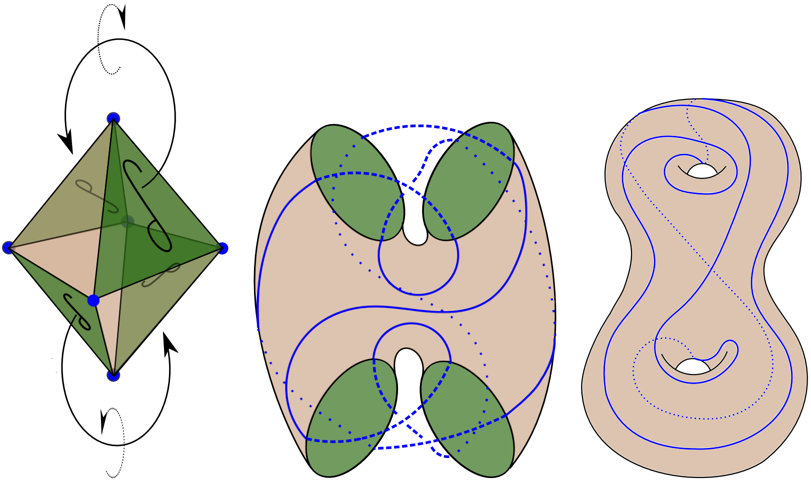

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The boundary $\Sigma$ is a four-holed sphere.

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Consider the regular ideal octahedron in $\mathbb{H}^3$, with vertices $\{0, \pm 1, \pm i, \infty\}$, and perform the indicated face identifications with Möbius transformations. This determines a representation $\rho_1 : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$.

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Since geometric finiteness is an open condition, we can deform the representation $\rho_1$ in $\mathcal{GF}(M)$. Let $\rho_t$ indicate the same face identifications, for the octahedron with vertices $\{0, \pm 1, \pm it, \infty\}$.

# A Path in $\mathcal{GF}(M)$

Consider the regular ideal octahedron in $\mathbb{H}^3$, with vertices $\{0, \pm 1, \pm i, \infty\}$, and perform the indicated face identifications with Möbius transformations. This determines a representation $\rho_1 : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$.

One may check:

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Let $\Gamma_t = \rho_t(\pi_1(\Sigma))$.

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**Question**

What is $\mathrm{Fix}\,\Phi$ in $\mathcal{T}(\Sigma)$?

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- $Ext(\xi, X) = 4Mod(Q_X)$, where Q is the quotient of $X$ by its two orientation-reversing involutions

# The Path of Quasi-Fuchsian Groups

**Lemma**

For $t \in (t_0, 1]$, $\Gamma_t$ is quasi-Fuchsian, with bending lamination on bottom (resp. top) given by $\theta_t \cdot \xi$ (resp. $\vartheta_t \cdot \eta$), and convex core boundary surface on bottom (resp. top) determined by $\ell_t = \ell(\xi, \rho_t)$ (resp. $\ell(\eta, \rho_t)$).

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$t_0$, $\theta_t$, and $\ell_t$ are all explicit.

Crucial step: Since $\Gamma_t$ is preserved by the rhombic symmetry $\Phi$, all of its geometric invariants are also. This ensures that the bending laminations are contained in the set $\{\xi, \eta\}$.

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**Problem**

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## Solution:

*Grafting* provides a geometric way of passing back and forth between the convex core boundary, with its bending lamination, and the conformal boundary:

$$\mathrm{gr} : \mathcal{ML}(\Sigma) \times \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma)$$

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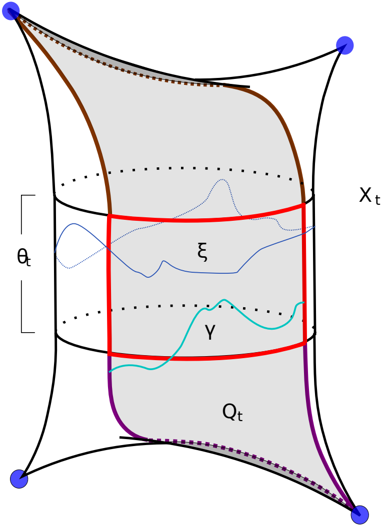


$X_t$ is $gr\,(\theta_t \cdot \xi,\, Y_t)$, where $Y_t$ is the $\{\xi, \eta\}$-rhombic 4-punctured sphere determined by $\ell(\xi, \Gamma_t)$.

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For non-monotonicity of $\sigma_M$, it suffices to show non-monotonicity of $Mod(Q_t)$

Explicit estimates on moduli of quadrilaterals are surprisingly hard, especially when the quadrilateral has an ideal vertex. Fortunately, a normalizing map will make a comparison accessible.

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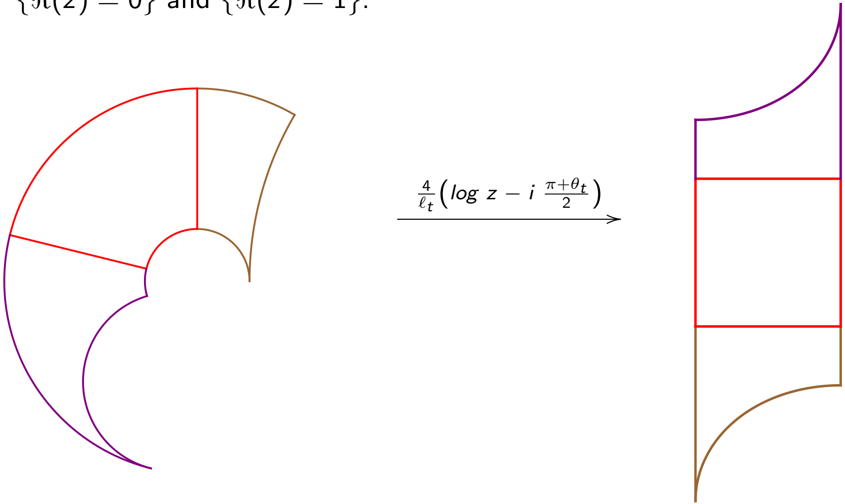

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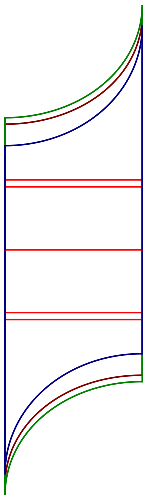

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- What is the critical point? An explanation of the geometric role of the symmetry?
- Families of skinning maps similar to this one? An understanding of the set of skinning maps obtained by picking rational points in the Masur domain of the genus-2 surface?