Hyperbolic Structures from Link Diagrams

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Motivation: knots

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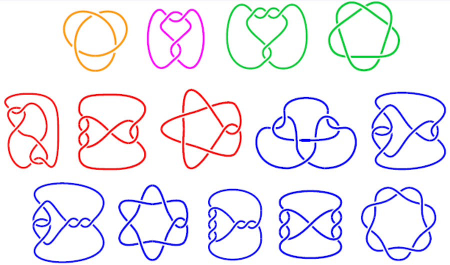

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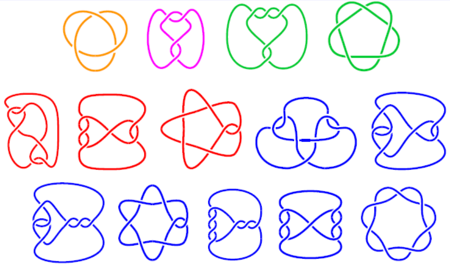

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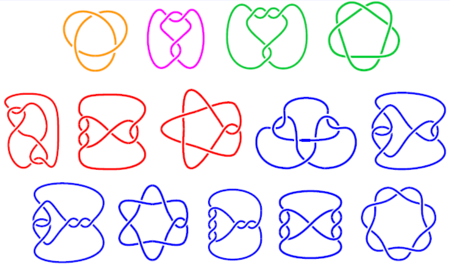

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Of the 1,701,935 prime knots up to 16 crossings, 32 are non-hyperbolic.
Of the 8,053,378 prime knots with 17 crossings, 30 are non-hyperbolic.

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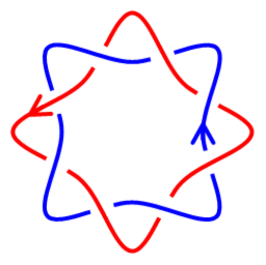

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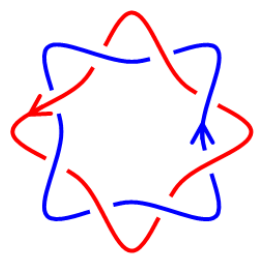
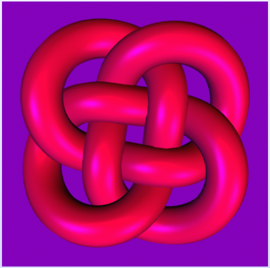
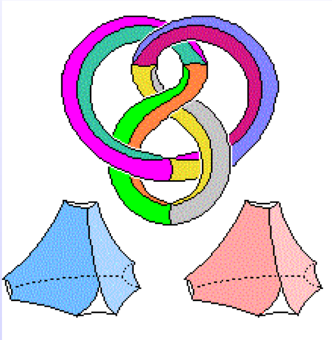
 

From the solution of the Tait flyping conjecture (W. Menasco, M. Thistlethwaite), each reduced alternating diagram of $(2, n)$–torus link is **standard**.

Methods for describing hyperbolic structure

A well-known method for describing the structure of hyperbolic manifolds by W. Thurston was implemented in the program SnapPea (J. Weeks). It is based on decomposition into ideal tetrahedra.



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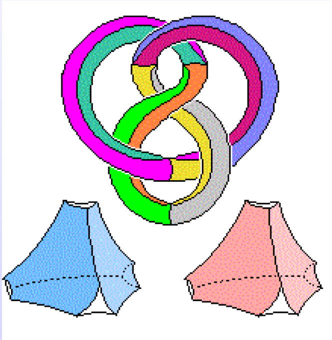
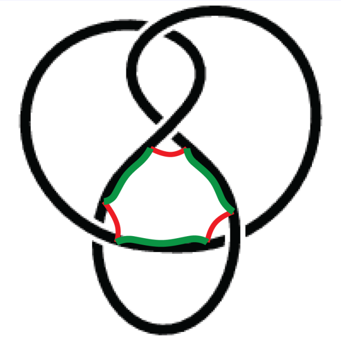
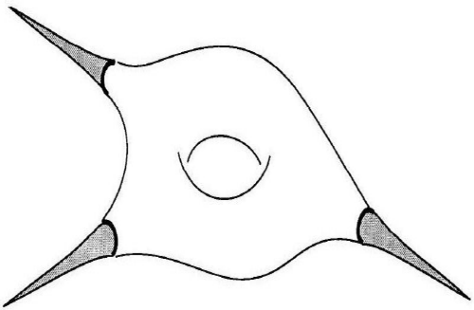




An **alternative method for links** is based on ideal polygons bounding the regions of a link diagram (M. Thistlethwaite).

Objectives

Consider a hyperbolic link complement. A preimage of a **cusp** in $\mathbb{H}^3$ is a set of horoballs.



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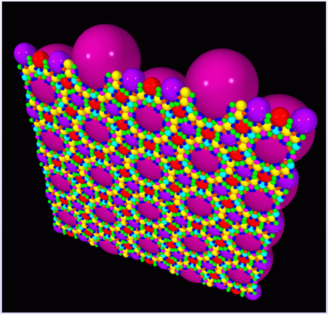

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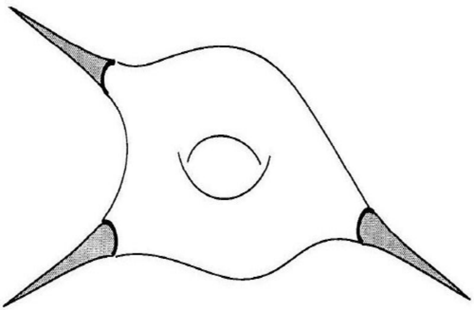

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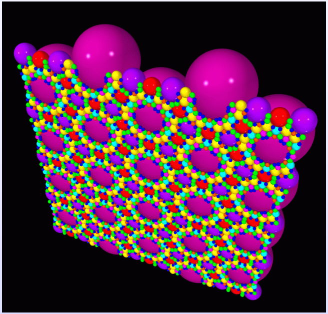

Horoball packing of figure-8 knot

Cusps may be chosen so that the horoballs have disjoint interiors. There are horoballs of arbitrarily small Euclidean diameter and one additional horoball, the plane $z = 1$. We want to describe **horoball packings** associated to complements of hyperbolic links.

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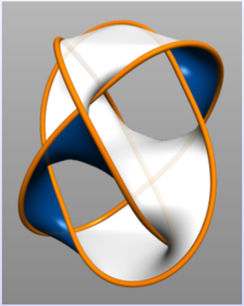

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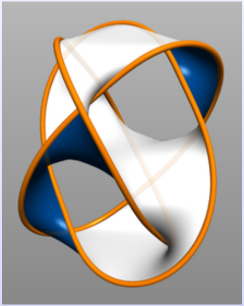

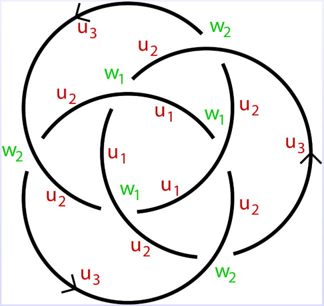

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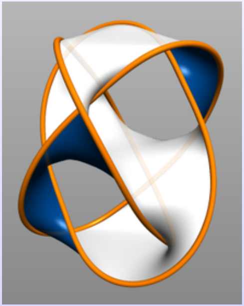

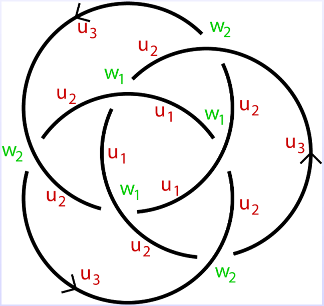

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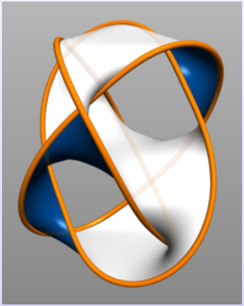

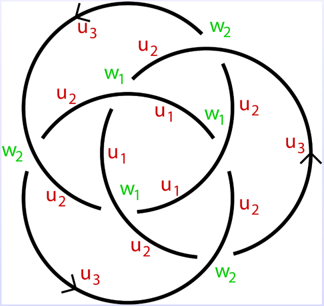

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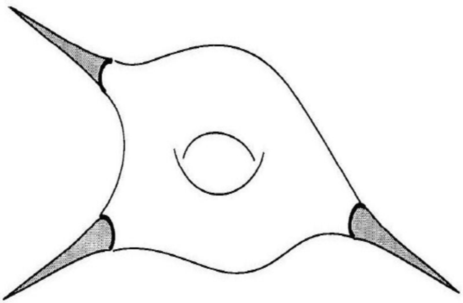
Take horospherical cross-sections of the cusps of $S^3 - L$, so that length of a meridian on each cross-section is 1.



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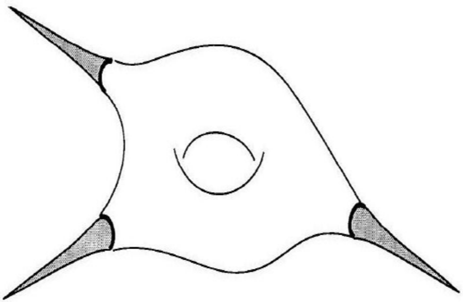

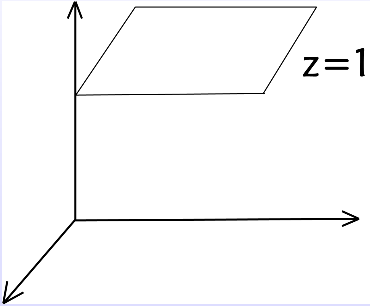

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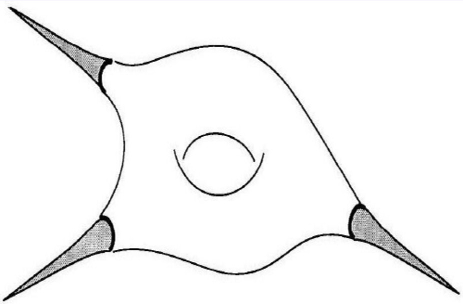
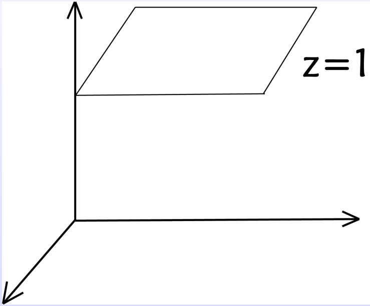
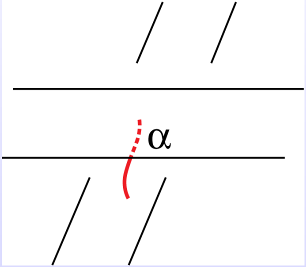




Choose the coordinates in $\mathbb{H}^3$ so that a component of the preimage of some particular cross-section is the Euclidean plane $z = 1$. Parameterize Euclidean translations on each horosphere by complex numbers so that the meridional translation corresponds to the real number 1.

Construction: crossing label w

Let α be an arc traveling vertically from overpass to underpass at a crossing of the diagram. It is properly homotopic to a unique geodesic in the link complement, which lifts to a geodesic γ in $\mathbb{H}^3$, joining the centers of the corresponding horospheres.



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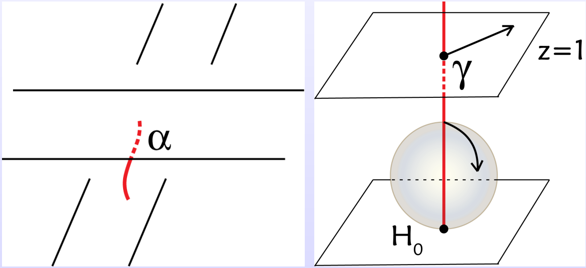

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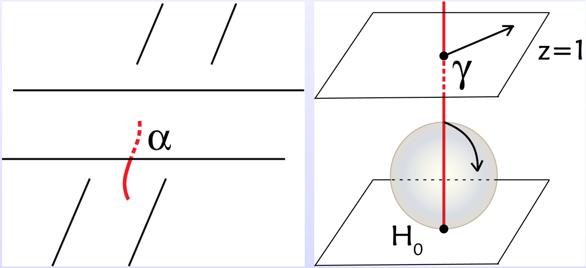
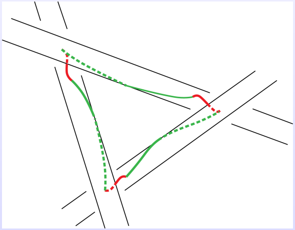


In $\mathbb{H}^3$ let $|w| = e^{-d}$, where d is the hyperbolic intercusp distance along the geodesic. Let the argument of w be the exterior dihedral angle between two half-planes, each determined by γ and the meridional translation on the one of the horospheres.

Construction: edge label u

A k-sided **region in the link diagram** is a disk whose boundary is a union of k arcs on the boundary torus and k arcs at crossings. The preimage of this boundary in $\mathbb{H}^3$ determines a cyclic sequence of horospheres $H_1, \dots, H_k$ with centers $P_1, \dots, P_k$.



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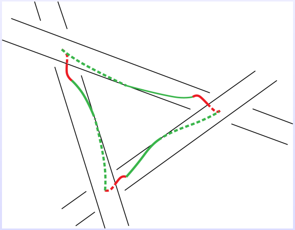

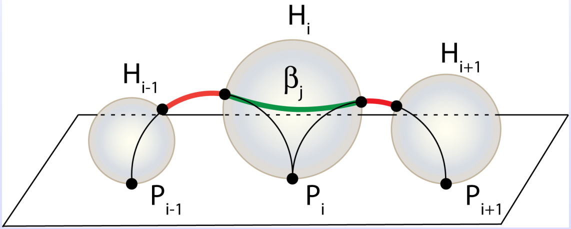

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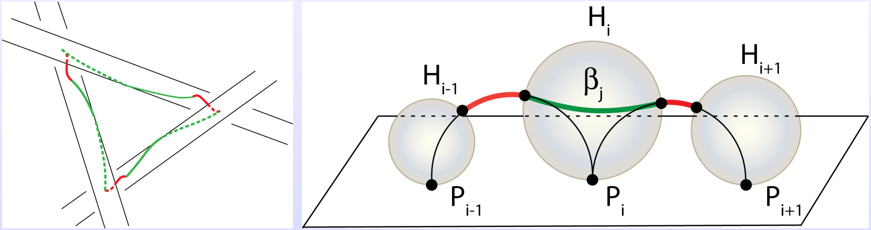

Let β_j be a sub-arc corresponding to a Euclidean line segment which joins the point where geodesics $P_{i-1}P_i$ and P_iP_{i+1} pierce H_i. The Euclidean translation taking the initial point of β_j to the terminal point defines u. Its orientation is inherited from the orientation of the link.

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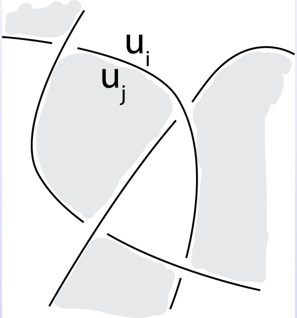
Color the regions of the link diagram in black and white as a checkerboard. Each edge gives rise to two arcs: on the boundary of the black region, and on the boundary of the white one.



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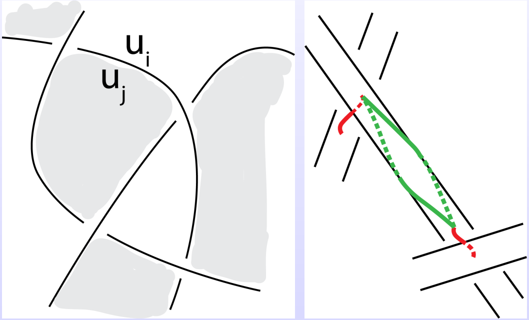
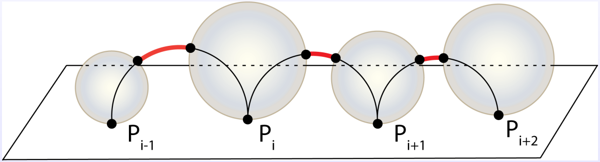


In an alternating diagram $u_i - u_j = 1$ holds for every edge. In a non-alternating diagram this difference may be 1, -1 or 0

Construction: shape parameters

Return to the ideal polygon with the vertices $P_1, ..., P_k$ in the boundary of $\mathbb{H}^3$.



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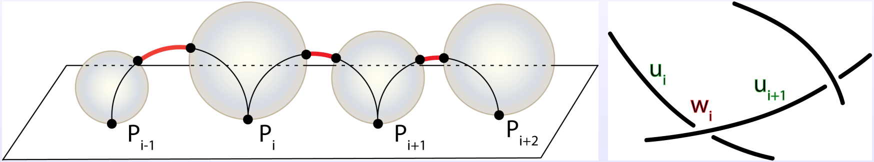

Define the **shape parameter** ξ_i of the geodesic $P_i P_{i+1}$ to be the cross-ratio

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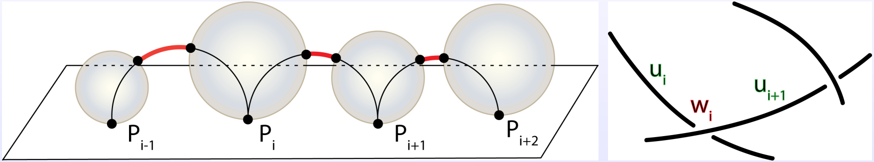

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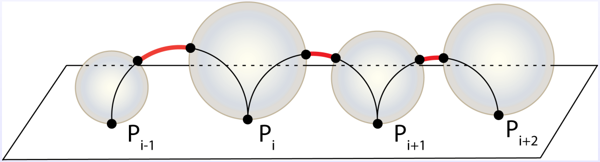

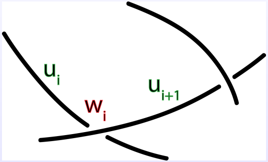

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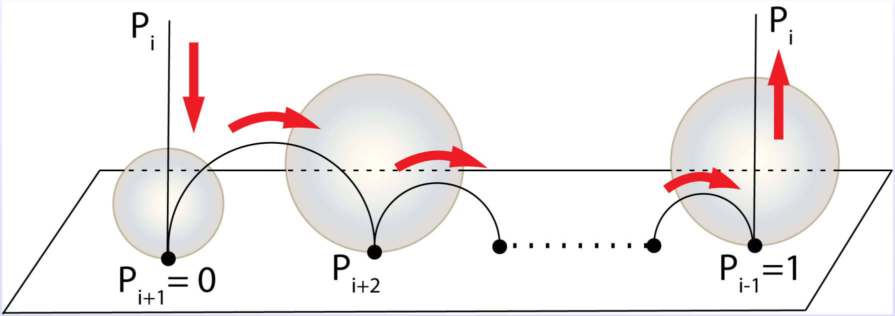
For a 3–sided region all $\xi_i = 1$. For a general k–sided region **we will obtain convenient relations for** ξ_i.

Relations

Perform an isometry to place the vertices P_{i-1}, P_i, P_{i+1} of the the polygon at $1, \infty, 0$ respectively.



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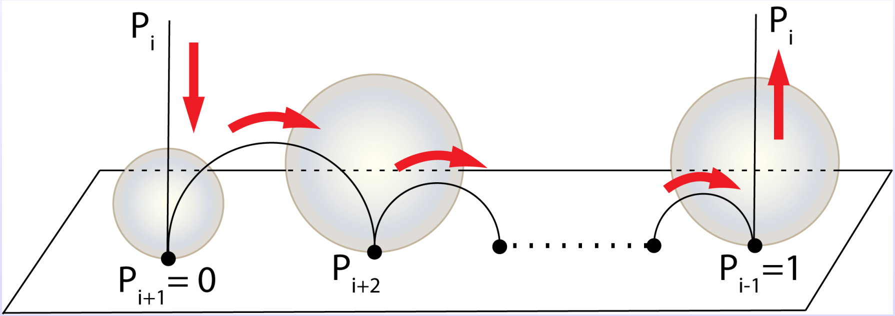

Then the Möbius transformation $\rho_i : z \to \frac{-\xi_i}{z-1}$ determines an isometry of $\mathbb{H}^3$ which maps P_{i-1}, P_i, P_{i+1} to P_i, P_{i+1}, P_{i+2} respectively.

Relations

Since the polygon closes up, the composite $\rho_k \circ ... \circ \rho_1 = 1$. If we represent the Möbius transformations by 2×2 matrices, we see that the product

$$\begin{pmatrix} 0 & -\xi_k \\ 1 & -1 \end{pmatrix} ... \begin{pmatrix} 0 & -\xi_1 \\ 1 & -1 \end{pmatrix}$$

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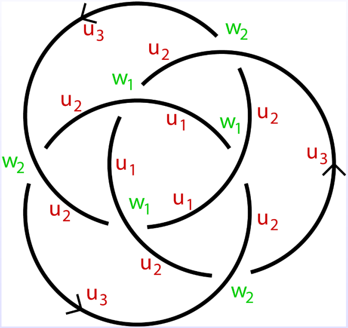
From the matrix entries we read off **3 independent polynomial relations for every region**. *E. g.*, for a 5–sided region

$$1 - \xi_i - \xi_{i+1} - \xi_{i+2} + \xi_i \xi_{i+2} = 0,\ 1 \leq i \leq 3.$$

Example: the Borromean Rings

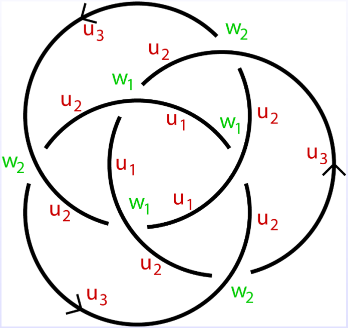


Recall that for 3–sided regions the shape parameters are 1, so

$$\frac{w_1}{u_1^2} = \frac{w_1}{u_2^2} = \frac{w_1}{-(u_1+1)(u_2+1)} = \frac{w_2}{(u_1+1)^2} = \frac{w_1}{(u_1+1)^2} = \frac{-w_2}{u_1 u_3} = 1. \text{ Hence,}$$

$$u_1 = u_2 = u_3 = \frac{1}{2}(-1+i),\ w_1 = -\frac{i}{2} = -w_2.$$

Example: the Borromean Rings



Recall that for 3–sided regions the shape parameters are 1, so

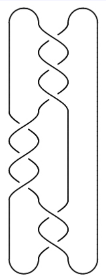
$$\frac{w_1}{u_1^2} = \frac{w_1}{u_2^2} = \frac{w_1}{-(u_1+1)(u_2+1)} = \frac{w_2}{(u_1+1)^2} = \frac{w_1}{(u_1+1)^2} = \frac{-w_2}{u_1 u_3} = 1. \text{ Hence,}$$

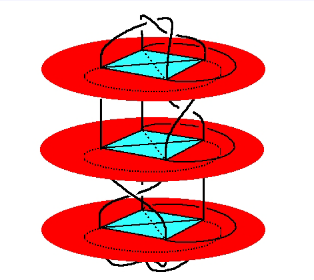
$$u_1 = u_2 = u_3 = \frac{1}{2}(-1+i),\; w_1 = -\frac{i}{2} = -w_2.$$

Computer calculations can be used in the systematic study.

Some applications: “exact” volume of 2–bridge links

Formulas that allow one to calculate **hyperbolic volume from a link diagram** were obtained using Sakuma-Weeks description of triangulation and the method. We obtain a polynomial, and the volume is expressed as a function of one of its roots (*i.e.*, the **volume is exact**).





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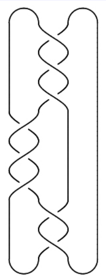

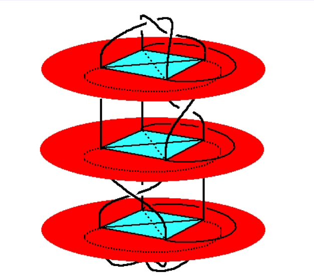

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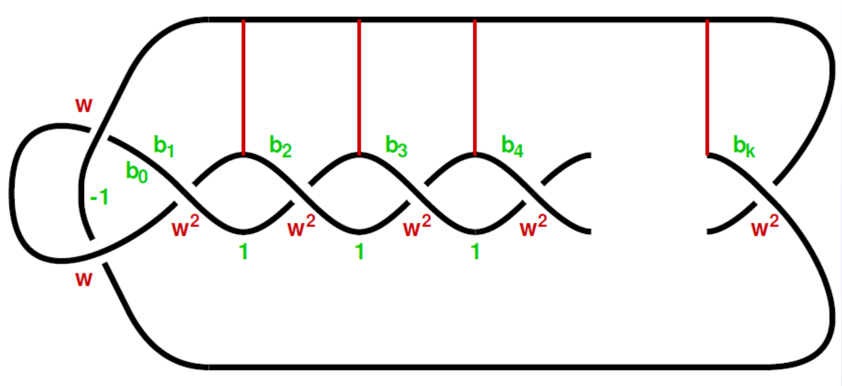
Same idea and formulas by C. Zickert can be used to compute the **complex volume** (its real part is hyperbolic volume, and the imaginary part is the Chern-Simons invariant).

Example: twist knot

In a twist knot with $k+2$ crossings, there are $k-1$ isometric pairs of tetrahedra. Their shape parameters are the ratios $z_i = \frac{b_i}{b_{i-1}}$.



All b_i can be written in terms of one label w. One easily obtains a polynomial for w from the rightmost region.

Some applications: link group and invariant fields

Edge and crossing labels determine a representation of a link group into $\mathrm{PSL}_2(\mathbb{C})$. The matrices for Wirtinger generators can be computed from the labels.

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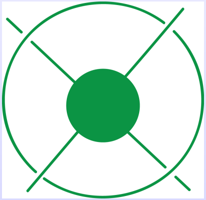
In a hyperbolic link the following tangle is called an **encircled tangle**. The labels on the circle are called **boundary labels**, and all the other - **interior labels**.



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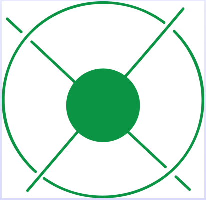

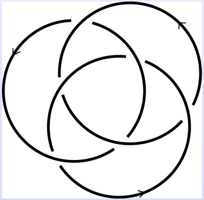

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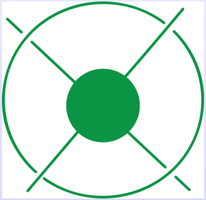

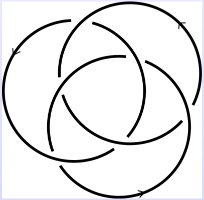

Theorem. Let (S^3, L) be a hyperbolic link containing an oriented encircled tangle (B, T). The interior labels of T are independent of the hyperbolic link L containing T.

Labels on tangles: the idea of proof

Suppose (B, T) is a tangle and $B - T$ admits a complete hyperbolic structure. Then the moduli space of complete hyperbolic structures on $B - T$ is connected and has real dimension two.

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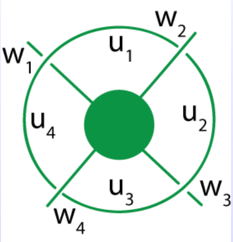

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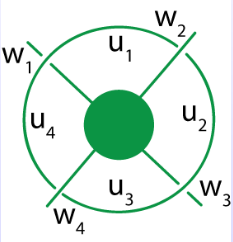

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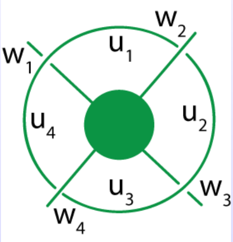

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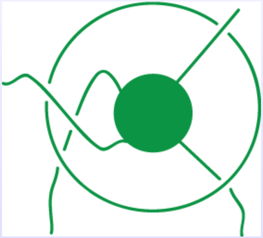
Labels on tangles: underlying geometry

The geometric reason for the rigidity is a 3–punctured sphere S in the complement of a tangle.



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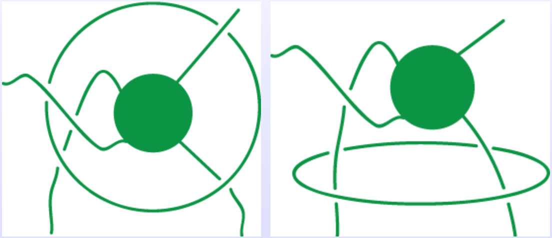

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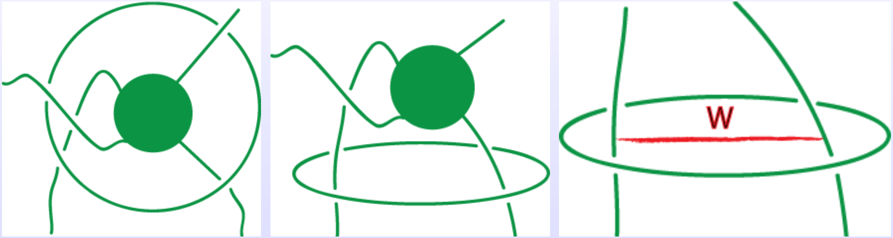
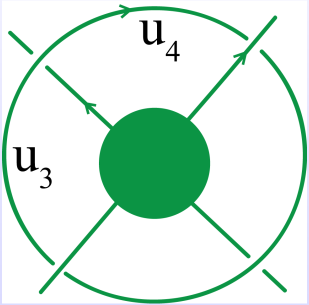


3–punctured sphere is a totally geodesic space (C. Adams). If we retract its cusps so that their boundaries have length 1, then $w = \pm\frac{1}{4}$, where the sign depends on the link orientation. This imposes an extra constraint of real dimension 2, determining the interior labels uniquely.

Labels on Tangles: Tangle Ratio

Corollary. The ratio $\frac{u_4}{u_3}$ does not change, being a numerical invariant of a tangle.



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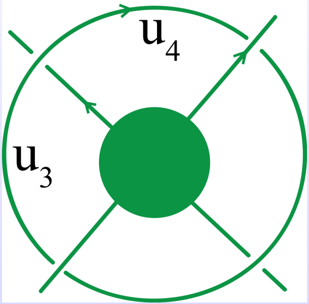

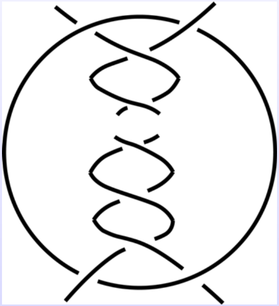

We obtained an equation for the ratio of a tangle with k twists. It can be generalized to an arbitrary rational tangle.

Related Questions: Intrinsic Geometry of Alternating Links

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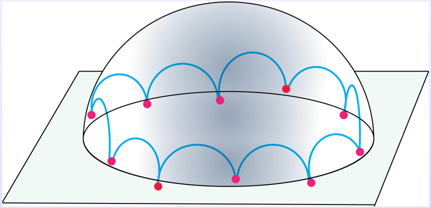

Conjecture 1. The preimage of a region of a reduced alternating link diagram is an ideal polygon in $\mathbb{H}^3$. Such polygon does not deviate far from being planar and regular.

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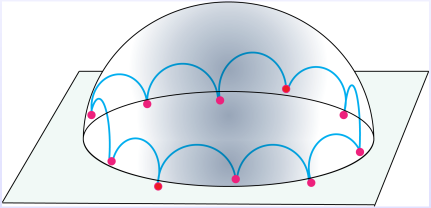
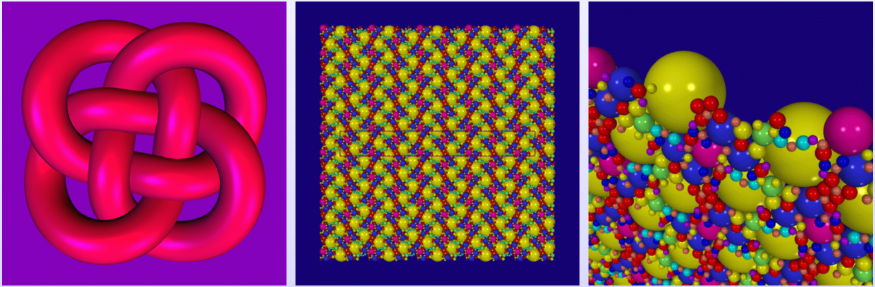


Conjecture 1. The preimage of a region of a reduced alternating link diagram is an ideal polygon in $\mathbb{H}^3$. Such polygon does not deviate far from being planar and regular.

Conjecture 2. For intercusp lengths of geodesics corresponding to crossings of alternating diagram, an upper bound is $\log 8$.

Questions

Horoball packing of Turk's Head knot.



Pictures of horoball packings by M. Thistlethwaite