Spectral networks

w/ D. Gaiotto, G. Moore – originally motiv. by physics, but seems to have some relation to some of the topics here

Fix:
- compact complex curve C
- $\vec{\varphi} = (\varphi_2, \dots, \varphi_K)$ with φ_r meromorphic section of $K_C^{\otimes r}$

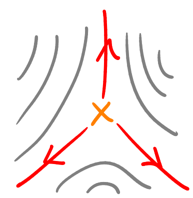
  $(\varphi_r = f(z)\, dz^r)$

$K=2$:

Singular foliation $F(\varphi_2)$ of C: leaves are paths where $\sqrt{\varphi_2}$ is real.

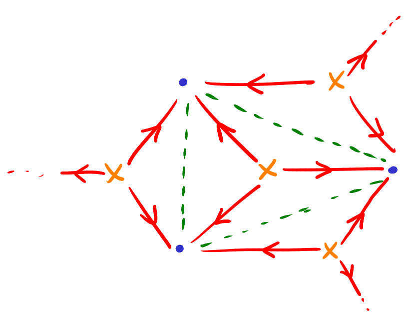
Assume φ_2 has only simple zeroes.

Then around each zero, $F(\varphi_2)$ looks like:



Let $W(\varphi_2)$ be the union of the red paths. ("critical graph of φ_2")

Assume φ_2 has at least one pole of order ≥ 2, and take φ_2 generic.

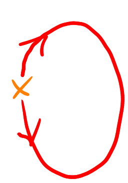


Then $W(\varphi_2)$ decomposes C into cells.

($\rightsquigarrow$ ideal triangulation of C.)

Topology of $W(\varphi_2)$ jumps at non-generic φ_2, where we have special leaves: in codim 1, the possibilities are

saddle connection



closed traj.

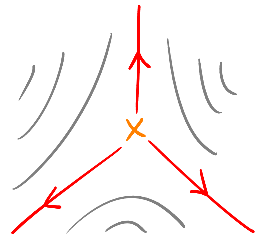
$K \geq 2$:

$P(\lambda, z) = \lambda^K + \sum_{r=2}^{K} \lambda^{K-r} \varphi_r(z)$. Locally on C, let $\lambda_i(z)$, $i = 1, \dots, K$, be the roots of $P(\lambda, z)$.

Local foliations $F_{ij}(\vec{\varphi})$ of C: leaves are paths where $\lambda_i - \lambda_j$ is real

Assume roots of $\Delta(z) = \operatorname{disc} P(z)$ all simple.

Then, around a root where $\lambda_i = \lambda_j$, $F_{ij}(\vec{\varphi})$ looks like

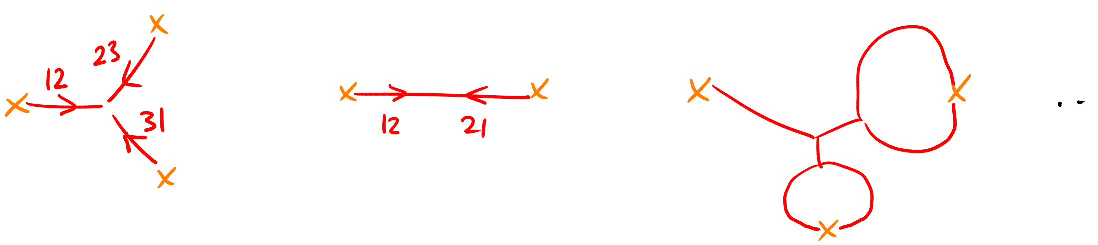


Now shoot critical trajectories from each zero of $\Delta(z)$.
Oriented, locally labeled by ij s.t. $\lambda_i - \lambda_j$ is real positive along the trajectory.

When two $\cap$ and have common index, they spawn a third:

(Show example.)

Topology <u>jumps</u> at nongeneric $\vec{\varphi}$ where some new objects appear:

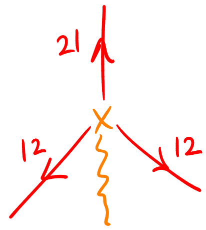


These objects appear to be a good higher-rank generaliz^{n} of the special traj. of quad diff.

<u>Coordinate systems</u>

$$\begin{array}{lll}
\Sigma = \{P(\lambda,z)=0\} & & \nabla^{ab} \in \mathcal{M}^b(\Sigma, GL(1)) = \{GL(1)\text{-conn. over } \Sigma\}/\sim \;\simeq (\mathbb{C}^\times)^{\#} \\
{\scriptstyle K:1}\big\downarrow & & \big\downarrow \qquad \big\downarrow \Psi_{\mathcal{W}} \\
C & & \nabla \in \mathcal{M}^b(C, GL(K)) = \left\{\begin{array}{l} GL(K)\text{-conn over } \Sigma \\ \text{plus flag data} \end{array}\right\}/\sim
\end{array}$$

∇ is <u>not</u> just the pushforward of ∇^{ab} — that would have <u>monodromy</u> around the branch points. Instead, <u>cut + glue</u> along the network $\mathcal{W}$ to <u>eliminate</u> this monodromy.



$$\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

Ψ_W is a symplectomorphism $\Longrightarrow$

Gives local Darboux coords $\mathcal{X}_\gamma^W$ on $\mathcal{M}^b(C, GL(K))$ labeled by classes $\gamma \in H_1(\Sigma, \mathbb{Z})$

Particular examples of this recover:

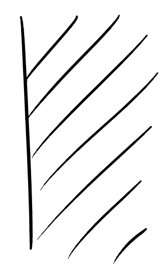
- Fock-Goncharov coordinates
- Fenchel-Nielsen
- Goldman

$\Big\}$ WIP w/ Dimofte-Hollands

Asymptotics

A pt. of $\mathcal{M}^b(C, SL(K))$ gives canonical 1-param family of flat connections $\nabla(\zeta)$ and also a canonical $\vec{\varphi}$. [Hitchin]

Hence a particular W.



$$\mathcal{X}_\gamma^W(\nabla(\zeta)) \sim \exp\left(\frac{\oint \lambda}{\zeta}\right)$$ as $\zeta \to 0$ in the right $\frac{1}{2}$-plane.

This is the key e.g. to a new description of the hyperkähler structure of $\mathcal{M}$...