Plumbing Constructions in Quasifuchsian Space.

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Dehn–Thurston coordinates

Given a pants decomposition $\mathcal{P} = \{\sigma_1, \ldots, \sigma_\xi\}$ on a surface Σ, Dehn defined an injection $\mathbf{i} : \mathcal{S} = \mathcal{S}(\Sigma) \longrightarrow \mathbb{Z}^{\xi}_{\geqslant 0} \times \mathbb{Z}^{\xi}$ by $\mathbf{i}(\gamma) = (q_1(\gamma), \ldots, q_\xi(\gamma); \mathrm{tw}_1(\gamma), \ldots, \mathrm{tw}_\xi(\gamma))$.

1. $q_i(\gamma) = i(\gamma, \sigma_i) \in \mathbb{Z}_{\geqslant 0}$ are the **length parameters**;
2. $\mathrm{tw}_i(\gamma) \in \mathbb{Z}$ are the **twist parameters** of γ.

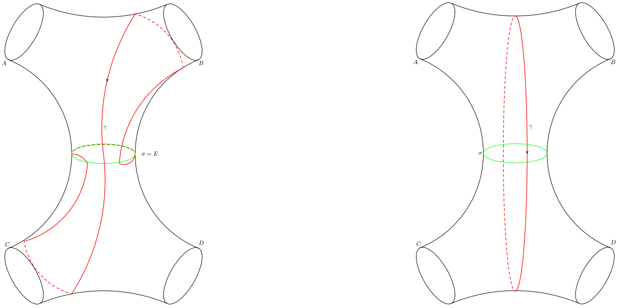


Figure: Penner and Harer twist $\hat{p}_i = -1$ and D. Thurston's twist $p_i = 0$.

Relation between $\hat{p}_i$ and p_i

Suppose two pairs of pants meet along $\sigma = E \in \mathcal{PC}$. Label their respective boundary curves (A, B, E) and (C, D, E) in clockwise order.

Theorem (M–Series)

Let $\gamma \in \mathcal{S}$ and let $\hat{p}_i$ and p_i denote the PH–twist and the DT–twist around σ. Then

$$\hat{p}_i = \frac{p_i + I(A, E; B) + I(C, E; D) - q_i}{2},$$

where $I(X, Y; Z)$ denotes the number of strands of $\gamma \cap P$ running from the boundary curve X to the boundary curve Y in the pair of pants $P = (X, Y, Z)$.

Thurston's symplectic form

Let τ_{Th} be Thurston symplectic form on $\mathcal{S} \subset \mathrm{ML}_{\mathbb{Q}}(\Sigma)$.

Theorem (M.)

Suppose that loops $\gamma, \gamma' \in \mathcal{S}$ belongs to the same chart and let $\mathbf{i}(\gamma) = (q_1, \ldots, q_\xi; p_1, \ldots, p_\xi), \mathbf{i}(\gamma') = (q'_1, \ldots, q'_\xi; p'_1, \ldots, p'_\xi)$ their DT coordinates. Then

$$\tau_{\mathrm{Th}}(\gamma, \gamma') = \frac{1}{2} \sum_{i=1}^{\xi} (q_i p'_i - q'_i p_i).$$

In addition, if γ, γ' are disjoint, then $\tau_{\mathrm{Th}}(\gamma, \gamma') = 0$.

Basic definitions on Kleinian groups

$\mathrm{PSL}(2,\mathbb{C})$ acts on $\mathbb{H}^3$ by isometries and on $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ by conformal maps.

Definition

- A **Kleinian group** G is a discrete (torsion-free) subgroup of $\mathrm{PSL}(2,\mathbb{C})$.
- The **limit set** $\Lambda(G)$ is the set of accumulation points of the action of G on $\hat{\mathbb{C}}$.
- The **regular set** $\Omega(G)$ is $\hat{\mathbb{C}} - \Lambda(G)$.
- A **Fuchsian group** is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$, or, equivalently, a Kleinian group G such that $\Lambda(G)$ is a circle.
- A **Quasifuchsian group** is a Kleinian group G such that $\Lambda(G)$ is a topological circle, or, equivalently, a quasi-conformal deformation of a Fuchsian group.

The Maskit embedding

The **Maskit slice** $\mathcal{M}$ is the set of representations $\rho : \pi_1(\Sigma) \longrightarrow PSL(2,\mathbb{C})$ (up to conjugation in $PSL(2,\mathbb{C})$) such that:

1. $G_\rho = \rho\left(\pi_1(\Sigma)\right)$ is discrete and ρ is an isomorphism,
2. $\rho(\sigma_i)$ are parabolic,
3. all components of $\Omega(G)$ are simply connected and there is exactly one invariant component $\Omega^+(G)$,
4. $\Omega^+(G)/G$ is homeomorphic to Σ and the other components are triply punctured spheres.

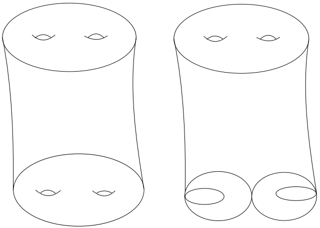


Figure: Quasifuchsian Group and Maskit Group.

Picture of the Maskit embedding for the once punctured torus $\Sigma_{1,1}$

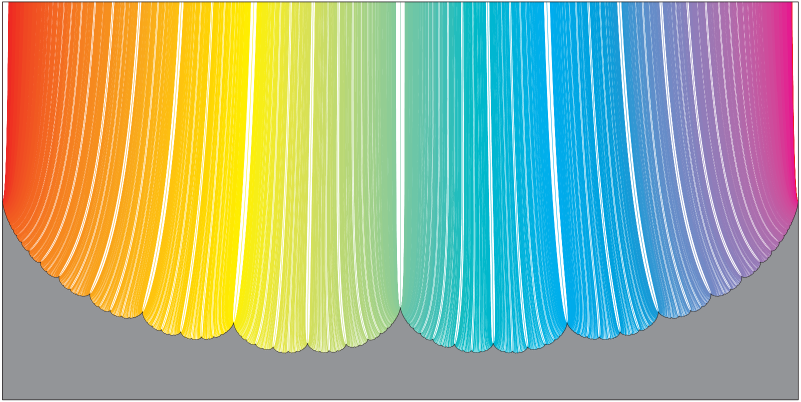
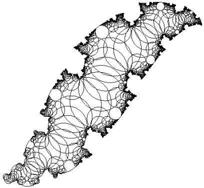
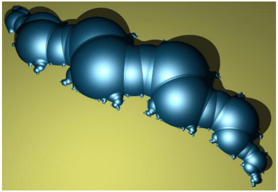


Figure: The Maskit embedding $\mathcal{M}(\Sigma_{1,1})$ for the once punctured torus. Picture courtesy David Wright.

Pleating Ray

A **pleated surface** is a a hyperbolic surface which is totally geodesic almost everywhere and such that the locus of points where it fails to be totally geodesic is a geodesic lamination.

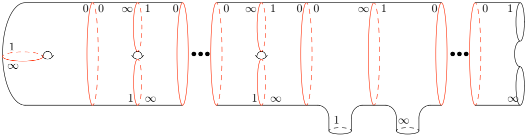




By Thurston, each component of the boundary $\partial\mathcal{C}(G)/G$ of the **convex core** is a pleated surface.
Given $\rho \in \mathcal{M}$, denote $\beta(\rho) \in \mathrm{ML}(\Sigma)$ the bending lamination of $\partial\mathcal{C}^+/G_\rho$, where $G_\rho = \rho(\pi_1(\Sigma))$.
Given $[\eta] \in \mathrm{PML}(\Sigma)$, the **pleating ray** $\mathcal{P} = \mathcal{P}_{[\eta]}$ of $[\eta]$ is the set of elements $\rho \in \mathcal{M}$ for which $\beta(\rho) \in [\eta]$.

Gluing construction

Let Σ be a surface with $\chi(\Sigma) < 0$ and let $\mathcal{PC} = \{\sigma_1, \ldots, \sigma_\xi\}$ be a pants decomposition on it. Let $\underline{\mu} = (\mu_1, \ldots, \mu_\xi) \in \mathbb{H}^\xi$.



STEP 1: Any triply punctured sphere is isometric to $\mathbb{P} = \mathbb{H}/\Gamma$, where

$$\Gamma = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle.$$

Identify any P_i to the fundamental domain Δ of Γ by the homeomorphisms

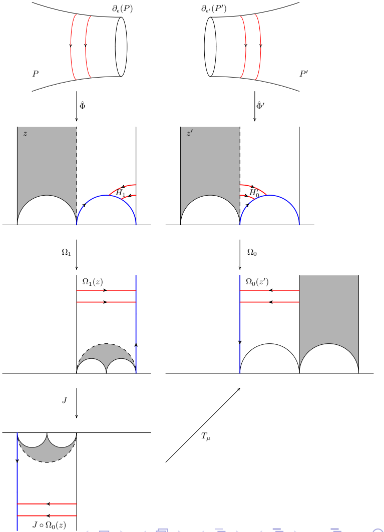
$$\Phi_i : \mathrm{int}(P_i) \longrightarrow \Delta.$$

Gluing construction

STEP 2: Let $\sigma_i = \partial_\epsilon P \cap \partial_{\epsilon'} P'$, then the gluing is described by

$$\Omega_\epsilon^{-1} J^{-1} T_{\mu_i}^{-1} \Omega_{\epsilon'}$$

where $\mu_i \in \mathbb{H}$ is the **gluing parameter** and $\Omega_\infty = \mathrm{Id}$,
$\Omega_0 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\Omega_1 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$,
$J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, $T_{\mu_i} = \begin{pmatrix} 1 & \mu_i \\ 0 & 1 \end{pmatrix}$.



Projective structure

This describes a (complex) projective structure on Σ, which depends on the gluing parameter $\underline{\mu} = (\mu_1, \dots, \mu_\xi) \in \mathbb{H}^\xi$. In particular, given $\underline{\mu} \in \mathbb{H}^\xi$, we define a *developing map* $\mathrm{Dev}_{\underline{\mu}} \colon \tilde{\Sigma} \longrightarrow \hat{\mathbb{C}}$ and a *holonomy map* $\rho_{\underline{\mu}} \colon \pi_1(\Sigma) \longrightarrow PSL(2, \mathbb{C})$.

Theorem (M–Series)

If $\mathrm{Dev}_{\underline{\mu}} \colon \tilde{\Sigma} \longrightarrow \hat{\mathbb{C}}$ is an embedding, then $\rho_{\underline{\mu}}$ is a group isomorphism and $\rho_{\underline{\mu}} \in \mathcal{M}$.

In addition, these representations $\rho_{\underline{\mu}}$ parametrise $\mathcal{M}$.

Top Terms' Formula

Let $\rho_{\underline{\mu}} \colon \pi_1(\Sigma) \longrightarrow PSL(2,\mathbb{C})$ be the holonomy described by the gluing construction. Let γ be a simple closed curve on Σ, not parallel to any of the pants curves σ_i.

Theorem (Top Terms' Formula, M – Series)

$$\operatorname{Tr} \rho_{\underline{\mu}}(\gamma) = \pm i^q 2^h \left(\mu_1 + \frac{(p_1 - q_1)}{q_1}\right)^{q_1} \cdots \left(\mu_\xi + \frac{(p_\xi - q_\xi)}{q_\xi}\right)^{q_\xi} + R,$$

where

- $q = \sum_{i=1}^{\xi} q_i > 0$;
- R *represents terms with total degree in* $\mu_1 \cdots \mu_\xi$ *at most* $q - 2$;
- $h = h(\gamma)$ *is the total number of waves.*

Asymptotic direction of pleating rays

Theorem (Asymptotic direction, M, Series, Keen–Series)

Suppose that $\eta = \sum_{i=1}^{\xi} a_i \gamma_i$ *is an* **admissible** *measured lamination on* Σ. *Then, as the bending measure* $\beta(G_\mu) \in [\eta]$ *tends to zero, the pleating ray* $\mathcal{P}_{[\eta]}$ *in* $\mathcal{M}$ *approaches the line*

$$\Re \mu_i = -\frac{p_i(\eta)}{q_i(\eta)} + 1, \qquad \frac{\Im \mu_1}{\Im \mu_j} = \frac{q_j(\eta)}{q_1(\eta)},$$

where $(q_1(\eta), \ldots, q_\xi(\eta); p_1(\eta), \ldots, p_\xi(\eta))$ *are the Dehn–Thurston coordinates for* η.

Generalised gluing construction

Given a pants decomposition $\mathcal{PC} = \{\sigma_1, \ldots, \sigma_\xi\}$ on Σ, let $\underline{c} = (c_1, \ldots, c_\xi) \in \mathbb{R}_+^\xi$ and $\underline{\boldsymbol{\mu}} = (\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_\xi) \in (\mathbb{C}/2i\pi)^\xi$.
We describe a (complex) projective structure on Σ with developing map $\mathrm{Dev}_{\underline{c},\underline{\boldsymbol{\mu}}}$ and *holonomy map* $\rho_{\underline{c},\underline{\boldsymbol{\mu}}}$. In particular, $\rho_{\underline{c},\underline{\boldsymbol{\mu}}}(\gamma)$ is hyperbolic and $\mathrm{Tr}\,\rho_{\underline{c},\underline{\boldsymbol{\mu}}}(\gamma) = \pm 2\cosh(c_j)$.

Theorem (M.)

If $\underline{c} \longrightarrow \underline{0}$ keeping $\underline{\mu}$ fixed, where $\mu_i = \frac{i\pi - \boldsymbol{\mu}_i}{c_i}$, then

$$\rho_{\underline{c},\underline{\boldsymbol{\mu}}} \longrightarrow \rho_{\underline{\mu}}.$$

Linear slices $\mathcal{L}_c$

Given $\mathcal{PC} = \{\sigma_1, \ldots, \sigma_\xi\}$, the *complex Fenchel–Nielsen coordinates* $\mathrm{FN}_\mathbb{C} \colon \mathcal{QF}(\Sigma) \longrightarrow (\mathbb{C}_+/2i\pi)^\xi \times (\mathbb{C}/2i\pi)^\xi$ are defined by

$$\mathrm{FN}_\mathbb{C}(G) = (\lambda_1, \ldots, \lambda_\xi, \tau_1, \ldots, \tau_\xi),$$

where λ_i are the *complex length* and τ_i are the *complex twist* of the pants curve σ_i.

Definition

Given $\underline{c} \in \mathbb{R}^\xi_+$, we define the *$\underline{c}$–slice* (or the *linear slice*) $\mathcal{L}_{\underline{c}}$ to be the set

$$\mathcal{L}_{\underline{c}} = \{(\underline{c}, \underline{\tau}) \in \mathrm{FN}_\mathbb{C}\left(\mathcal{QF}(\Sigma)\right) \mid \mathrm{sign}(\Im \tau_1) = \ldots = \mathrm{sign}(\Im \tau_\xi)\}.$$

Connectedness of linear slices $\mathcal{L}_c$

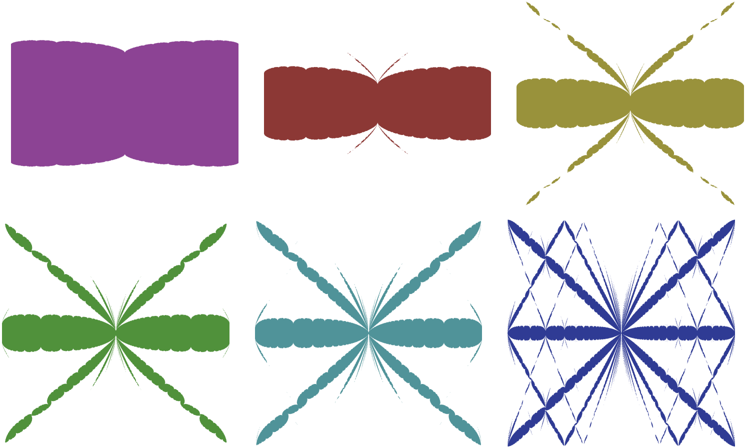
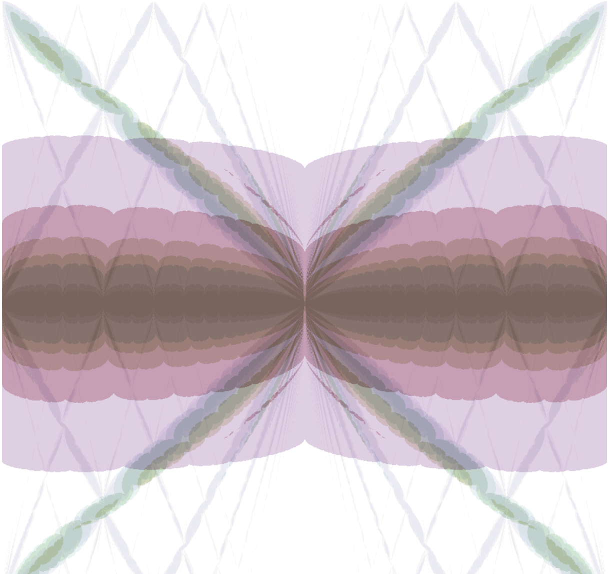


Figure: The linear slice $\mathcal{L}_c$ when $c = 1, 2, 4, 5, 10, 20$.

Connectedness of linear slices $\mathcal{L}_c$



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