Higgs bundles over elliptic curves

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• Joint work with O. Garcia-Prada and P. Newstead.

• PhD *Higgs bundles over elliptic curves*, study Higgs bundles for different structure groups:

- Classical complex Lie groups: $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{PGL}(n,\mathbb{C})$, $\mathrm{Sp}(2m,\mathbb{C})$, $\mathrm{O}(n,\mathbb{C})$ and $\mathrm{SO}(n,\mathbb{C})$;
- Real forms of $\mathrm{GL}(n,\mathbb{C})$: $\mathrm{U}(p,q)$, $\mathrm{U}^*(2m)$ and $\mathrm{GL}(n,\mathbb{R})$;
- G arbitrary connected complex reductive Lie group.

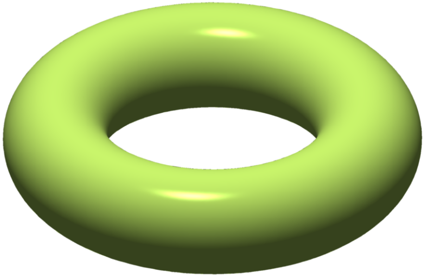
• In this talk we focus on $G = \mathrm{GL}(n,\mathbb{C}) \Longrightarrow$ We work with **vector bundles**.

• **Objective of the talk: EXPLICIT DESCRIPTION of the moduli space of Higgs bundles over an elliptic curve** with zero degree.

• First explicit description of a moduli space of Higgs bundles.

Elliptic curves

- An elliptic curve is a Riemann surface of genus 1.



- X elliptic curve
  - using Abel-Jacobi we obtain $\text{Jac}(X) \cong X$ (the Jacobean $\text{Jac}(X)$ is the space of all line bundles over X);
  - $\text{Jac}(X)$ abelian group structure (with $\otimes$) $\Longrightarrow$ abelian group structure on X;
  - the canonical bundle is trivial $K_X \cong \mathcal{O} \cong X \times \mathbb{C}$;
  - also the cotangent bundle is trivial $T^*X \cong X \times \mathbb{C}$.

Moduli space of vector bundles

- We need 3 things for a moduli problem:
  - $A =$ **collection** of holomorphic vector bundles $\mathcal{E} \to X$ (ideally all, but not possible)
  - $\sim$ **relation** between vector bundles $\mathcal{E} \sim \mathcal{E}'$ (ideally $\sim$ would be $\cong$, but no possible)
  - $\mathbb{E} \to X \times Y$ **family** of vector bundles parametrized by Y (vector bundle over $X \times Y$).

- A *(coarse) moduli space M* is a variety such that
  - as a set is equal to $^{A}/_{\sim}$ (there exists a bijection $^{A}/_{\sim} \xrightarrow{1:1} M$)
  - its algebraic structure is given by the families (any family $\mathbb{F} \to X \times Y$ defines an algebraic morphism $\nu_{\mathbb{F}} : Y \to M$)
  - it has some universality property correpresenting this functor

$$Y \longmapsto \left\{ \begin{array}{c} (\sim)\text{-equivalence classes of} \\ \text{families parametrized by } Y \end{array} \right\}$$

- Is impossible to construct a (nice, haussdorf) moduli space of vector bundles unless
  - Collection $\Longrightarrow$ A is the **collection of all semistable** vector bundles
  - Relation $\Longrightarrow$ $\sim$ is **S-equivalence**

Theorem (Mumford, Seshadri, Narasimham)

There exists a moduli space $M(n,d)$ of S-equivalence classes of semistable vector bundles over a compact Riemann surface.

(Semi)stability and S-equivalence

• A holomorphic vector bundle $\mathcal{E}$ is **semistable** if for every subbundle $\mathcal{F} \subset \mathcal{E}$

$$\frac{\deg \mathcal{F}}{\operatorname{rk} \mathcal{F}} \leq \frac{\deg \mathcal{E}}{\operatorname{rk} \mathcal{E}}$$

• $\mathcal{E}$ is **stable** if the above inequality is always strict ($\leq \implies <$)
• $\mathcal{E}$ is **polystable** if

$$\mathcal{E} \cong \bigoplus_i \mathcal{E}_i, \qquad \text{with all } \mathcal{E}_i \text{ stable vector bundles.}$$

• (Using the Jordan-Hölder filtration) to **any semistable** $\mathcal{E}$ vector bundle we can associate a **unique (up to isomorphism) polystable** vector bundle

$$\mathcal{E} \text{ semistable} \implies \operatorname{gr}(\mathcal{E}) \text{ polystable} \qquad (\text{associated } \textbf{graded object} \text{ of } \mathcal{E}).$$

• $\mathcal{E}$ and $\mathcal{E}'$ semistable vector bundles are **S-equivalent** if $\operatorname{gr}(\mathcal{E}) \cong \operatorname{gr}(\mathcal{E}')$.

• **On every S-equivalence class there is always a unique polystable** (up to isomorphism).

$$\text{Semistability} \big/ \text{S-equivalence} \iff \text{Polystability} \big/ \text{isomorphism} .$$

Atiyah: Vector bundles over an elliptic curve

• [Atiyah 1957] *Vector bundles over elliptic curves*. Beautiful paper before GIT and moduli theory (see [Tu 1993] for stability considerations).
• **Atiyah gives a complete description** of vector bundles of any rank and degree **but**, to simplify this talk, **we will focus on the case of degre** $d = 0$.

Proposition (Atiyah)

(Over an elliptic curve) the only stable vector bundles of degree 0 *are those of rank* 1 *(line bundles* $\mathcal{L} \in \mathrm{Jac}(X)$*).*

Corollary

(Over an elliptic curve) $\mathcal{E}$ *is a polystable vector bundle of degree* 0 *if and only if* $\mathcal{E} \cong \bigoplus_i \mathcal{L}_i$*, where* $\mathcal{L}_i \in \mathrm{Jac}(X)$.

• $\mathbb{L}$ family of all line bundles of degree 0 parametrized by $X (\cong \mathrm{Jac}(X))$.
• $\mathbb{E}$ **family of all polystable bundles** parametrized by $X \times \overset{n}{\dots} \times X$ (n copies of $\mathbb{L}$)
• Moduli theory $\Longrightarrow$ $\mathbb{E}$ induces a morphism $(X \times \overset{n}{\dots} \times X) \to M(n, 0)$. It factors through

$$\mathrm{Sym}^n X \xrightarrow{\cong} M(n, 0).$$

A priori this is only a bijective morphism, it is an **isomorphism** since $M(n, 0)$ is smooth (see LePotier's book).

- A **Higgs bundle** over the elliptic curve X a pair $(\mathcal{E}, \varphi)$, where
  - $\mathcal{E}$ is a **vector bundle** over X
  - the Higgs field φ is an **endomorphism** of the vector bundle ($\varphi \in H^0(X, \operatorname{End}\mathcal{E})$) (when $g > 1$ $\varphi \in H^0(X, \operatorname{End}\mathcal{E} \otimes K_X)$ but recall that K_X is trivial in genus $g = 1$)

- Moduli problem fo Higgs bundles:
  - $\mathcal{A}$ = collection of semistable Higgs bundles of rank n and degree d over elliptic curve X.
  - $\sim$ = S-equivalence
  - **Usual definition of families** $\widetilde{\mathbb{F}} = (\mathbb{F}, \Phi)$ where $\mathbb{F} \to X \times Y$ family of vector bundles and $\Phi \in H^0(X \times Y, \operatorname{End}\mathbb{F})$.

Theorem (Hitchin, Simpson, Nitsure)

There exists a coarse moduli space of Higgs bundles $\mathcal{M}(n, d)$ for the above moduli problem.

- The Higgs bundle $(\mathcal{E}, \varphi)$ is
  - **semistable** if for any subbundle $\mathcal{F}$ such that $\varphi(\mathcal{F}) \subset \mathcal{F}$
$$\frac{\deg \mathcal{F}}{\operatorname{rk} \mathcal{F}} \leq \frac{\deg \mathcal{E}}{\operatorname{rk} \mathcal{E}},$$
  - **stable** if the above inequality is strict ($<$) for any subbundle $\mathcal{F}$ such that $\varphi(\mathcal{F}) \subset \mathcal{F}$
  - **polystable** if
$$(\mathcal{E}, \varphi) \cong \bigoplus_i (\mathcal{E}_i, \varphi_i), \qquad \text{where } (\mathcal{E}_i, \varphi_i) \text{ are stable Higgs bundles.}$$

- (Using the Jordan-Hölder filtration) to **any semistable** $(\mathcal{E}, \varphi)$ Higgs bundle we can associate a **unique (up to isomorphism) polystable** Higgs bundle

$$(\mathcal{E}, \varphi) \text{ semistable } \Longrightarrow \operatorname{gr}(\mathcal{E}, \varphi) \text{ polystable} \qquad (\text{associated } \textbf{graded object} \text{ of } (\mathcal{E}, \varphi)).$$

- $(\mathcal{E}, \varphi)$ and $(\mathcal{E}', \varphi')$ semistable Higgs bundles are **S-equivalent** if $\operatorname{gr}(\mathcal{E}, \varphi) \cong \operatorname{gr}(\mathcal{E}', \varphi')$.

- **On every S-equivalence class there is always a unique polystable** (up to isomorphism).

$$\text{Semistability} \Big/ \text{S-equivalence} \Longleftrightarrow \text{Polystability} \Big/ \text{isomorphism}.$$

- We are lucky:

Proposition

(Over an elliptic curve) $(\mathcal{E}, \varphi)$ *semistable Higgs bundle* $\Longleftrightarrow$ $\mathcal{E}$ *semistable vector bundle.*

- Very lucky indeed:

Proposition

(Over an elliptic curve) $(\mathcal{E}, \varphi)$ *stable Higgs bundle* $\Longleftrightarrow$ $\mathcal{E}$ *stable vector bundle.*

- Everything is so easy:

Corollary

(Over an elliptic curve) a Higgs bundle of degree 0 $(\mathcal{E}, \varphi)$ *is polystable if and only if*

$$(\mathcal{E}, \varphi) \cong \bigoplus_{i=1}^{n} (\mathcal{L}_i, \phi_i), \qquad \mathcal{L}_i \in \mathrm{Jac}(X) \quad \text{and} \quad \phi_i \in H^0(\mathrm{End}\, \mathcal{L}_i) \cong H^0(\mathcal{O}) \cong \mathbb{C}.$$

- $\widetilde{\mathbb{L}}$ family of **all line Higgs bundles** parametrized by $T^*X \cong X \times \mathbb{C} \cong \mathrm{Jac}(X) \times H^0(\mathcal{O})$.
- $\widetilde{\mathbb{E}}$ family of **all polystable Higgs bundles** of deg $= 0$ parametrized by $T^*X \times \overset{n}{\ldots} \times T^*X$ (n copies of $\widetilde{\mathbb{L}}$).
- Moduli theory $\Longrightarrow$ $\widetilde{\mathbb{E}}$ induces a morphism $(T^*X \times \overset{n}{\ldots} \times T^*X) \longrightarrow \mathcal{M}(n,0)$. It factors through a bijective morphism

$$\mathrm{Sym}^n\, T^*X \xrightarrow{1:1} \mathcal{M}(n,0)$$

- *If the target of a bijection is normal, it is an isomorphism* (by Zariski's Main Theorem).
- But...

First attempt

- $\widetilde{\mathbb{L}}$ family of **all line Higgs bundles** parametrized by $T^*X \cong X \times \mathbb{C} \cong \mathrm{Jac}(X) \times H^0(\mathcal{O})$.
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- *If the target of a bijection is normal, it is an isomorphism* (by Zariski's Main Theorem).
- But... **normality of $\mathcal{M}(n,0)$ for elliptic curves ($g=1$) is an open question**. (For $g > 1$ we know that $\mathcal{M}(n,0)$ is normal).

• A family $\widetilde{\mathbb{V}}$ parametrized by Z has the **local universal property** if for **any other family** $\widetilde{\mathbb{F}}$ parametrized by Y and any $y \in Y$, there exists $y \in U \subset Y$ open and $f : U \to Z$ such that

$$f^* \widetilde{\mathbb{V}} \sim \widetilde{\mathbb{F}}|_U.$$

• From Moduli theory: *Suppose $\widetilde{\mathbb{V}}$ parametrized by Z has the local universal property, Γ group acting on Z such that $\widetilde{\mathbb{V}}_{z_1} \sim \widetilde{\mathbb{V}}_{z_2} \Longleftrightarrow z_2 = \gamma \cdot z_1$ for some $\gamma \in \Gamma$. Then, a categorical quotient of Z by Γ is a coarse moduli space if and only if it is an orbit space.*

• Using $\widetilde{\mathbb{E}}$ we have $Z = T^*X \times \overset{n}{\ldots} \times T^*X$ and $\Gamma = \mathfrak{S}_n$ the symmetric group. Since $\mathfrak{S}_n$ is finite $Z/\Gamma = \mathrm{Sym}^n\, T^*X$ is always an orbit space.

• **Problem: $\widetilde{\mathbb{E}}$ doesn't have the local universal property** (too many families $\widetilde{\mathbb{F}}$).

A new moduli problem

- **We change the moduli problem:**
  - $\mathcal{A}$ = collection of semistable Higgs bundles (unchanged)
  - $\sim$ = S-equivalence ($(\mathcal{E}_1, \varphi_1) \sim (\mathcal{E}_2, \varphi_2) \Longleftrightarrow \mathrm{gr}(\mathcal{E}_1, \varphi_1) \cong \mathrm{gr}(\mathcal{E}_2, \varphi_2)$) (unchanged)
  - **New definition of families.** A familly of Higgs bundles $\widetilde{\mathbb{F}} = (\mathbb{F}, \Phi)$ parametrized by Y is **locally graded** if for every $y \in Y$ there exists $y \in U \subset Y$ open and $\widetilde{\mathbb{L}}_1, \ldots, \widetilde{\mathbb{L}}_n$ families of line Higgs bundles, such that

$$\widetilde{\mathbb{F}}|_U \sim \bigoplus_{i=1}^{n} \widetilde{\mathbb{L}}_i.$$

Proposition

$\widetilde{\mathbb{E}}$ *has the local universal property among locally graded families.*

Theorem

There exists a coarse moduli space $\mathcal{N}(n,d)$ for the new moduli problem, and

$$\mathcal{N}(n,0) \cong \mathrm{Sym}^n T^*X.$$

There exists a bijection $\mathcal{N}(n,0) \xrightarrow{1:1} \mathcal{M}(n,0)$ and furthermore $\mathcal{N}(n,0)$ is the normalization of $\mathcal{M}(n,0)$.

Hitchin fibration

• Hitchin 1987, evaluating the invariant homogeneus polynomials $P_1, \dots, P_n$ on the Higgs field

$$h : \mathcal{M}(n,d) \longrightarrow B = \bigoplus_i H^0(X, K^{\otimes \deg P_i})$$

$$(\mathcal{E}, \varphi) \longmapsto (P_1(\varphi), \dots, P_n(\varphi))$$

• in our case

$$\begin{array}{ccc} \mathcal{N}(n,d) & \xrightarrow{\quad h \quad} & B = \bigoplus_i H^0(X, \mathcal{O}) (\cong \mathbb{C}^n) \\ \Big\downarrow \cong & & \cong \Big\downarrow \text{using the } P_i \\ \operatorname{Sym}^n T^*X & \xrightarrow{\quad p \quad} & \operatorname{Sym}^n \mathbb{C} \end{array}$$

• $p^{-1}\left([\lambda_1, \overset{m_1}{\dots}, \lambda_1, \dots, \lambda_\ell, \overset{m_\ell}{\dots}, \lambda_\ell]_{\mathfrak{S}_n}\right) \cong \operatorname{Sym}^{m_1} X \times \cdots \times \operatorname{Sym}^{m_\ell} X$, so

Corollary

The generic Hitchin fibre (all $m_i = 1$) is the abelian variety $X \times \overset{n}{\dots} \times X$. The non-generic fibre is a holomorphic fibration over the abelian variety $X \times \overset{\ell}{\dots} \times X$ with fibre $\mathbb{P}^{(m_1-1)} \times \overset{\ell}{\dots} \times \mathbb{P}^{(m_\ell-1)}$.

Nice picture

- By the result on semistability, there exists a projection $\mathcal{N}(n,d) \xrightarrow{a} M(n,d)$.
- Two important maps

$$\begin{array}{ccc} & \mathcal{N}(n,0) & \\ {}^{a}\swarrow & & \searrow^{h} \\ M(n,0) & & B \end{array}$$

- easy in our case

$$\begin{array}{ccc} & \operatorname{Sym}^n T^*X & \\ \swarrow & & \searrow \\ \operatorname{Sym}^n X & & \operatorname{Sym}^n \mathbb{C} \end{array}$$

- Let $\widetilde{M}(n,0)$ and $\widetilde{\mathcal{N}}(n,0)$ be the orbifolds given by the quotients $(X\times \overset{n}{\cdots} \times X)\,/\,\mathfrak{S}_n$ and $(T^*X\times \overset{n}{\cdots} \times T^*X)\,/\,\mathfrak{S}_n$, we have that

$$\widetilde{\mathcal{N}}(n,0) \cong \mathcal{T}^*\widetilde{M}(n,0)$$

where $\mathcal{T}^*$ denotes the cotangent orbifold bundle.