Hyperbolic geometry in higher Teichmüller theory

Bill Dreyer

University of Southern California/Notre Dame

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Purpose of this talk

Let S be a closed, connected, oriented surface with $\chi(S) < 0$.

Consider the character variety

$$\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S) = \mathrm{Hom}\big(\pi_1(S), \mathrm{PSL}_n(\mathbb{R})\big)//\mathrm{PSL}_n(\mathbb{R})$$

when $n \geq 2$.

We are interested in studying *geometrically* representations lying in the **Hitchin space** $\mathcal{H}_{\mathrm{PSL}_n(\mathbb{R})}(S)$, which is the connected component of $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$ containing a copy of the **Teichmüller space** $\mathcal{T}(S)$.

Two (geometric!) approaches

i) **Anosov representation** (Labourie)

ii) **Positive representation** (Fock-Goncharov)

Both approaches yield the existence of a certain **equivariant boundary curve**.

Theorem (Labourie, Fock-Goncharov)

Let $\rho : \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ be a Hitchin representation. There exists a unique, positive, continuous ρ–equivariant flag curve

$$\mathcal{F}_\rho : \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n).$$

One property highly reminiscent of Teichmüller representations...

Theorem (Labourie, Fock-Goncharov)

Every Hitchin/Anosov representation $\rho : \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ *is discrete and injective.*

Furthermore, for every $\gamma \in \pi_1(S) - \{1\}$*,* $\rho(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$ *diagonalizes, its eigenvalues* $\lambda_i^\rho(\gamma)$ *are all real with distinct absolute values; we index them such that*

$$|\lambda_1^\rho(\gamma)| > \cdots > |\lambda_n^\rho(\gamma)|\,.$$

Outline

1. Length functions for Anosov representations
2. Cataclysm deformations of Anosov representations
3. Parametrizing the Hitchin space

Length functions for Anosov representations

Case $\mathbf{n = 2}$: Thurston's length function $\ell_{\mathbf{m}}$

Given $m \in \mathcal{T}(S)$, there exists a continuous, homogeneous function

$$\ell_m : \mathcal{ML}(S) \to \mathbb{R}^+$$

where $\mathcal{ML}(S)$ is the space of *measured laminations on* S.

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Interesting features

Very nice (full) invariant of $m \in \mathcal{T}(S)$, which detects *earthquake/stretching/grafting deformations*, differentiability properties, can be used to parametrize $\mathcal{T}(S)$ via *shearing coordinates*, compactification of $\mathcal{T}(S)$, etc...

Case "n = n": Length functions ℓ_i^ρ

Let

$$\mathcal{C}^{\mathrm{Hol}}(S) = \left\{ \begin{array}{c} \text{Transverse Hölder distributions} \\ \text{for the geodesic foliation on } T^1 S \end{array} \right\}.$$

$\mathcal{C}^{\mathrm{Hol}}(S)$ is the (vector) space of *Hölder geodesic currents on S*.

Fact

$$\left\{ \text{conjugacy classes } \bar{\gamma} \text{ of } \pi_1(S) - \{1\} \right\} \subset \mathcal{C}^{\mathrm{Hol}}(S) \supset \mathcal{ML}(S)$$

Let $\rho : \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ be an Anosov representation.

Theorem (D.)

There exist n linear functions $\ell_i^\rho : \mathcal{C}^{\mathrm{Hol}}(S) \to \mathbb{R}$ such that for every closed geodesic $\bar{\gamma} \in \mathcal{C}^{\mathrm{Hol}}(S)$,

$$\ell_i^\rho(\bar{\gamma}) = \log \left| \lambda_i^\rho(\gamma) \right| .$$

Furthermore, for every current $\alpha \in \mathcal{C}^{\mathrm{Hol}}(S)$,

i) $\sum_{i=1}^n \ell_i^\rho(\alpha) = 0$;
ii) $\ell_i^\rho(R^*\alpha) = -\ell_{n-i+1}^\rho(\alpha)$

where $R : T^1S \to T^1S$ is defined the involution by $R(u) = -u$.

*The function ℓ_i^ρ is the i–**th length** of the Anosov representation ρ.*

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Californian deformations of Anosov representations

Let λ be a *maximal geodesic lamination* in S.

$S - \lambda$ is made of ideal triangles $\rightsquigarrow$ **"Ideal triangulation"** of S.

A **cataclysm** is a deformation consisting of shearing the ideal triangles contained in the complement $S - \lambda$ along the leaves of the geodesic lamination λ.

$\rightsquigarrow$ This modifies the **"gluing"** of the ideal triangles along λ.

"The Big One" Theorem

Let $\rho : \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ be an Anosov representation.

$\mathcal{C}^{\mathrm{twist}}(\widehat{\lambda})$ is the (vector) space of *n–twisted currents for* $\widehat{\lambda} \subset T^1S$.

Theorem (D.)

There exist an open neighborhood $0 \in \mathcal{U}^\rho \subset \mathcal{C}^{\mathrm{twist}}(\widehat{\lambda})$ *and a continuous, injective map*

$$\begin{aligned} \Lambda : \mathcal{U}^\rho &\to \mathcal{R}^{\mathrm{Anosov}}_{\mathrm{PSL}_n(\mathbb{R})}(S) \\ \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) &\mapsto \Lambda^\varepsilon \rho \end{aligned}$$

such that $\Lambda^0 \rho = \rho$.

The Anosov representation $\Lambda^\varepsilon \rho$ *is the* **cataclysm deformation** *of* ρ *of magnitude* $\varepsilon \in \mathcal{C}^{\mathrm{twist}}(\widehat{\lambda})$.

Some remarks

- The construction of cataclysms makes use of the associated flag curve $\mathcal{F}_\rho : \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$;

  $\rightsquigarrow$ **Flag decorated lamination** $\widetilde{\lambda}^{\mathcal{F}_\rho}$

- They should be understood as a **deformation of the flag curve** $\mathcal{F}_\rho : \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$;

- When $n = 3$: cataclysms include **bending deformations of real convex projective structures** on S introduced by B. Goldman.

Variation of the lengths ℓ_i^ρ

Let $\mathcal{C}^{\mathrm{Hol}}(\widehat{\lambda})$ be the (vector) space of *Hölder geodesic currents for* $\widehat{\lambda}$.

Let $\tau : \mathcal{C}^{\mathrm{Hol}}(\widehat{\lambda}) \times \mathcal{C}^{\mathrm{Hol}}(\widehat{\lambda}) \to \mathbb{R}$ be *Thurston's intersection number*.

Theorem (D.)

Let ℓ_i^ρ and $\ell_i^{\rho'}$ be the i–th lengths associated with ρ and $\rho' = \Lambda^\varepsilon \rho$, respectively. For every current $\alpha \in \mathcal{C}^{\mathrm{Hol}}(\widehat{\lambda})$,

$$\ell_i^{\rho'}(\alpha) = \ell_i^\rho(\alpha) + \tau(\alpha, \varepsilon_i).$$

$\rightsquigarrow$ The lengths ℓ_i^ρ **detect cataclysm deformations.**

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Parametrizing the Hitchin space

Using Higgs bundle techniques, N. Hitchin proved the following result.

Theorem (Hitchin)

The Hitchin space $\mathcal{H}_{\mathrm{PSL}_n(\mathbb{R})}(S)$ is homeomorphic to $\mathbb{R}^{(n^2-1)(2g-2)}$.

We would like to recover the above result using the *intrinsic geometry* of Hitchin representations.

This is a Higgs bundle free space.

Thurston-Fock-Goncharov's philosophy...

We would like to obtain a parametrization of the Hitchin space $\mathcal{H}_{\mathrm{PSL}_n(\mathbb{R})}(S)$ *"à la Thurston-Fock-Goncharov"*, where the first half of the coordinates are the **lengths** $\{\ell_i\}_i$ and the second half are some **triangle invariants** $\{X^T_{abc}\}$, namely

$$\mathcal{H}_{\mathrm{PSL}_n(\mathbb{R})}(S) \ni \rho \mapsto \Big(\underbrace{\ell_1^\rho , \ \dots \ , \ \ell_n^\rho}_{\text{Shear invariants}} \ , \underbrace{\{X^{T_1}_{abc}(\rho)\}, \dots, \{X^{T_{4g-4}}_{abc}(\rho)\}}_{\text{Triangle invariants}} \Big)$$

Flag decorated triangles

Let ρ be a Hitchin representation along with its flag curve $\mathcal{F}_\rho$.

Fact (Fock-Goncharov)

$$\mathcal{F}_\rho : \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n) \text{ is } \textbf{positive}.$$

In particular, for every $T \subset \widetilde{S} - \widetilde{\lambda}$, the **geometry of each flag decorated triangle** $T^{\mathcal{F}_\rho}$ is determined by $(n-1)(n-2)/2$ positive numbers $\{X^T_{abc}(\rho)\}_{a+b+c=n}$ called **triangle invariants**.

A careful analysis of the dimension

Fact (D.)

Cataclysms are deformations of dimension $(n-1)(6g-6)+\left\lfloor\frac{n-1}{2}\right\rfloor$ which **preserve the geometry of the flag decorated triangles**.

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- **Triangle dimension (expected):**

$$\underbrace{(4g-4)}_{\substack{\#\text{ triangles}\\ \text{in } S-\lambda}} \times \underbrace{\frac{(n-1)(n-2)}{2}}_{\substack{\#\text{ triangle invariants per}\\ \text{flag decorated triangle } T^{\xi}}}.$$

$$\textbf{Shear dim.} + \textbf{Triangle dim.} = (2g-2)(n^2-1) + \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$> \textbf{dim. of } \mathcal{H}_{\mathbf{PSL_n(\mathbb{R})}}\mathbf{(S)}$$

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Theorem (Bonahon, D.)

There are $\left\lfloor \frac{n-1}{2} \right\rfloor$ $\mathbb{Z}$–linear relations in the $\{\{\log X^T_{abc}\}\}_{T\subset S-\lambda}$.

As a result, the **triangle dimension** *is*

$$(4g-4)\times\frac{(n-1)(n-2)}{2} - \left\lfloor \frac{n-1}{2} \right\rfloor .$$

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- Differentiability properties $\rightsquigarrow$ asymptotic estimates;
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- Play a crucial rôle in analysing existing constraints on the triangle invariants $\{\{X_{abc}^{T}\}\}_{T\subset S-\lambda}$;

"À la Thurston-Fock-Goncharov-Bonahon-D."...

Let λ be a maximal lamination with finitely many leaves.

Theorem (Bonahon, D.)

The map

$$\begin{aligned}\Theta_\lambda : \mathcal{H}_{\mathrm{PSL}_n(\mathbb{R})}(S) &\to \mathcal{P}\\ \rho &\mapsto \left(\sigma_1^\rho, \ldots, \sigma_{n-1}^\rho, \{X_{abc}^{T_1}(\rho)\}, \ldots, \{X_{abc}^{T_{4g-4}}(\rho)\}\right)\end{aligned}$$

defines a homeomorphism from the Hitchin space $\mathcal{H}_{\mathrm{PSL}_n(\mathbb{R})}(S)$ onto **the interior of a convex polytope** *$\mathcal{P} \subset \mathbb{R}^N$.*

Morevover, $\mathcal{P}$ is a certain bundle over $\mathbb{R}^{(n-1)(n-2)(2g-2)-\lfloor \frac{n-1}{2} \rfloor}$, whose fibers are homeomorphic to $\mathbb{R}^{(n-1)(6g-6)+\lfloor \frac{n-1}{2} \rfloor}$.

The shear invariants σ_i^ρ are **quasi-currents** for $\widehat{\lambda}$.

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- More to come...

Thanks everyone!

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Please, once again for them: Plaudite cives!