Volume and rigidity for hyperbolic 3-manfolds

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Slogan: Volume is a topological invariant of hyperbolic 3-manifolds.

References:

W. Thurston, *The Geometry and Topology of* 3*-Manifolds*, http://www.msri.org/publications/books/gt3m

H. Munkholm, Springer Lecture Notes #788.

N. Dunfield, "Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds", *Invent. Math* **136**(1999)

C. Löh, "Measure homology and singular homology are isometrically isomorphic", *Math. Z.* **253** (2006).

Textbooks: Benedetti and Petronio, Ratcliffe

Definition (Gromov). If $x \in H_k(X;\mathbb{R})$ is a singular homology class, define

$$||x|| = \inf \left\{ \sum |a_i| : \sum a_i \sigma_i \in x \right\}.$$

This is a seminorm. If M is an orientable n-manifold, let $[M] \in H_n(M;\mathbb{R})$ denote its fundamental class, and define $||M|| = ||[M]||$.

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Pessimists might expect $||M||$ to always be 0, but clearly it is a *topological* invariant, and has these important properties:

- If $f : X \to Y$ and $x \in H_k(X)$ then $||x|| \geq ||f_*(x)||$.
- If M and N are orientable n-manifolds and $f : M \to N$ has degree n then $||M|| \geq n||N||$.

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- If M and N are orientable n-manifolds and $f : M \to N$ has degree n then $||M|| \geq n||N||$.

Theorem (Gromov). *If M is an orientable hyperbolic n-manifold then $||M|| = \mathrm{vol}\, M/v_n$ where v_n is the volume of a regular ideal hyperbolic n-simplex.*

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Definition. A *measure k-chain* on a CW-complex X is a compactly supported, signed Borel measure of bounded total variation on the space $C^1(\Delta_k, X)$.

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If we let $\mathcal{C}_k(X)$ denote the abelian group of measure k-chains, this gives an inclusion $\iota_k : C_k(X) \to \mathcal{C}_k(X)$.

Let $\phi_i : \Delta_{k-1} \to \Delta_k$ denote the face inclusions, which induce $\phi_i^* : C^1(\Delta_k, X) \to C^1(\Delta_{k-1}, X)$. For a measure k-chain μ, define

$$\partial_n(\mu) = \sum_i (-1)^i (\phi_i^*)_*(\mu).$$

This makes $\mathcal{C}_*$ into a chain complex and ι into a chain map. Write the homology groups as $\mathcal{H}_k$.

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Define the *total variation* of a measure-chain as

$$||\mu|| = \sup \left\{ \int g\mu : \; |g(\sigma)| \leq 1 \text{ for all } \sigma \in C^1(\Delta_k, X) \right\}.$$

Define a seminorm on $\mathcal{H}_k$ by $||x|| = \inf \{||\mu|| : \; \mu \in x\}$.

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Define a seminorm on $\mathcal{H}_k$ by $||x|| = \inf \{||\mu|| : \ \mu \in x\}$.

We still have:

- If $f : X \to Y$ and $x \in \mathcal{H}_k(X)$ then $||f_*(x)|| \leq ||x||$.

Theorem (C. Löh, 2006). *For any connected CW-complex X, the inclusion $\iota : C_*(X) \to \mathcal{C}_*(X)$ induces an isomorphism $H_k(X) \to \mathcal{H}_k(X)$ which is an isometry with respect to their seminorms.*

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We may identify $[M]$ with $\iota([M])$ and define $||M|| = ||[M]||$, where now $[M] \in \mathcal{H}_n(M)$. This is Thurston's definition of Gromov's norm, and the two definitions are equivalent by Löh's Theorem.

In particular,

- If M and N are orientable n-manifolds and $f : M \to N$ has degree n then $||M|| \geq n||N||$.

To work with measure chains we need to know that the usual pairing (i.e. integration) between k-chains and k-forms extends to measure chains. Here is the definition. If μ is a measure k-chain and ω is a k-form,

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- If μ and μ' are measure k-cycles then $[\mu] = [\mu']$ if and only if $\langle \mu, \omega \rangle = \langle \mu', \omega \rangle$ for all closed k-forms ω.
- If M is a Riemannian n-manifold and μ is a measure n-cycle then $[\mu] = (\langle \mu, dV \rangle / \operatorname{vol} M)[M]$.

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Suppose $\sigma : \Delta_k \to \mathbb{H}^n$ is a singular simplex. The points $\sigma(v_0), \ldots, \sigma(v_k)$ span a hyperbolic k-simplex Δ_σ. There is a canonical map $S(\sigma) : \Delta_k \to \Delta_\sigma$ which preserves barycentric coordinates. There is a homotopy from σ to $S(\sigma)$, constant on the vertices. ($S(\sigma)$ is the straightening of σ.)

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Suppose $\sigma : \Delta_k \to M$ is a singular simplex. Lift it to $\tilde{\sigma} : \Delta_k \to \mathbb{H}^n$. Define:

$$S(\sigma) = p \circ S(\tilde{\sigma}).$$

Let $\mathcal{S}_k(M) \subset \mathcal{C}_k(M)$ denote the measures supported on the image of S. Since S commutes with the boundary map, $\mathcal{S}_*(M)$ is a chain sub-complex. The map $\mu \to S_*(\mu)$ is a chain-homotopy inverse to the inclusion.

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We need the key geometrical fact that the regular ideal tetrahedron is the unique hyperbolic tetrahedron of maximal volume. (In $\mathbb{H}^3$ we have a volume formula. In $\mathbb{H}^n$ this is a theorem of Haagerup and Munkholm, proved after Gromov's proof for $n = 3$. So now we can do this for n-manifolds)

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But μ is supported on straight simplices, for which $\int_\sigma dV = \mathrm{vol}\, \sigma(\Delta_n) < v_n$. Since $\mathrm{vol}\, M < v_n ||\mu||$ for any $\mu \in \mathcal{S}_n(M)$ with $[\mu] = [M]$; it follows that $\mathrm{vol}\, M \leq v_n ||[M]||$.

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$$\gamma \cdot \sigma = p \circ \gamma \circ \tilde{\sigma}.$$

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If $\sigma : \Delta_k \to M$ is a singular simplex, define Smear(σ) to be the measure chain supported on the orbit of σ, with the measure which is locally the product of $\pm dV$ on an open set in M with the unit mass Haar measure on the $SO(n)$-fibers. (Use $+$ if σ is positively oriented, $-$ if it is negatively oriented.)

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In particular, $||\mathrm{Smear}(\sigma)|| = \mathrm{vol}\, M$ for *any* singular simplex σ.

Let $\sigma : \Delta_n \to M$ be a **straight** positive singular simplex and let v_σ denote the volume of its image hyperbolic simplex. If $\mu = \mathrm{Smear}(\sigma)$ then

$$\langle \mu, dV \rangle = \int \left(\int_\sigma dV \right) \mu = v_\sigma \int \mu = v_\sigma \operatorname{vol} M.$$

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Of course $\mathrm{Smear}(\sigma)$ is not a cycle. However, let $\bar{\sigma}$ be the simplex obtained by composing σ with reflection in one of its faces. (Orient $\bar{\sigma}$ negatively). Notice that each oriented face of σ is mapped to an oriented face of $\bar{\sigma}$ by a hyperbolic rotation.

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This implies $\Sigma = \text{Smear}(\sigma) - \text{Smear}(\bar{\sigma})$ is a cycle!

Moreover, since Σ is supported on two disjoint orbits, $||\Sigma|| = 2\, \mathrm{vol}\, M$. Since $\langle \Sigma, dV \rangle = (v_\sigma - (-v_\sigma))\, \mathrm{vol}\, M$, we have $[\Sigma] = 2v_\sigma [M]$. Therefore $||M|| \leq \frac{||\Sigma||}{2v_\sigma} = \frac{\mathrm{vol}\, M}{v_\sigma}$.

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But we may take v_σ arbitrarily close to v_n, so $\operatorname{vol} M \geq ||M|| v_n$.

Gromov used his theorem to give a simple proof of Mostow's Rigidity Theorem.

Theorem (Mostow). *Suppose* $M_1 = \mathbb{H}^n/\Gamma_1$ *and* $M_2 = \mathbb{H}^n/\Gamma_2$ *are closed orientable hyperbolic n-manifolds with* $n > 2$*. If* M_1 *is homotopy equivalent to* M_2 *then* Γ_1 *is conjugate to* Γ_2 *in* $Isom_+\mathbb{H}^n$*, and hence* M_1 *is isometric to* M_2*.*

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Start with a homotopy equivalence $f : M_1 \to M_2$. Lift f to $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$. Say an $n+1$-tuple of points on S^n_∞ is *regular* if they span a regular ideal n-simplex.

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- Show $\tilde{f}$ is a quasi-isometry, and deduce that $\tilde{f}$ extends continuously, giving $\tilde{f}_\infty : S^2_\infty \to S^2_\infty$. (See Munkholm.)
- Show that $\tilde{f}_\infty$ sends regular 4-tuples to regular 4-tuples.
- Show that this condition on $\tilde{f}_\infty$ implies that f is an isometry.

Suppose $\tilde{f}_\infty$ maps the vertices of a regular ideal simplex Δ to the vertices of an irregular ideal simplex Δ', with $\mathrm{vol}\,\Delta' < v_n - 2\epsilon$. Take a sequence σ_k of straight (non-ideal) simplices with vertices tending to the ideal vertices of Δ.

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Let $\mu_k = \frac{1}{2}\mathrm{Smear}(\sigma_k)$ and $\bar{\mu}_k = \frac{1}{2}\mathrm{Smear}(\bar{\sigma}_k)$. We know that $[\mu_k - \bar{\mu}_k] = (\int_{\sigma_k} dV)/v_n)[M_1]$, and hence that $[S_* f_*(\mu_k - \bar{\mu}_k)] \to [M_2]$ as $k \to \infty$.

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There is an open set $U \subset \mathrm{Isom}_+\mathbb{H}^n$ so that, for all $g \in U$, and all sufficiently large n, $\operatorname{vol} S \circ f(g \cdot \sigma_k) < v_n - \epsilon$.Thus

$$\langle S_* f_* \mu_k, dV\rangle < \mu(U)(v_n - \epsilon) + \left(\frac{1}{2}\operatorname{vol} M_1 - \mu(U)\right) v_n,$$

Since $\operatorname{vol} M_1 = \operatorname{vol} M_2$ by Gromov's theorem, this gives $\langle S_* f_*(\mu_k - \bar{\mu}_k), dV\rangle < v_n \operatorname{vol} M_2 - \mu(U)\epsilon$, contradicting that $[S_* f_*(\mu_k - \bar{\mu}_k)] \to [M_2]$.

We know that $\tilde{f}_\infty$ sends regular $(n+1)$-tuples to regular $(n+1)$-tuples. We will show that $\tilde{f}_\infty$ is a Möbius transformation. Since the action of Γ_i is determined by its action on S^2_∞, this implies that Γ_1 is conjugate to Γ_2 in $\mathrm{Isom}_+\mathbb{H}^n$.

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Let g be the Möbius transformation that agrees with $\tilde{f}_\infty$ on the vertices of Δ. Since $\tilde{f}_\infty$ takes regular $(n+1)$-tuples to regular $(n+1)$-tuples, f agrees with g on the vertices of each simplex in the the tesselation.

Thus $\tilde{f}_\infty$ agrees with g on a dense set of S^∞. Since $\tilde{f}_\infty$ is continuous, it is equal to g.

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Quiz: Where did we use $n > 2$?

Answer

In dimension $n > 2$ there are exactly 2 regular ideal n-simplexes having a given regular ideal $(n-1)$-simplex Φ as a face. The reflection through Φ takes one of these n-simplexes to the other. So there is a unique way to extend the regular ideal simplex Δ to a tesselation of $\mathbb{H}^n$ by regular ideal simplexes.

In dimension 2, every ideal 2-simplex is regular. There are uncountably many ways to extend the regular ideal simplex Δ to a tesellation of $\mathbb{H}^2$ by regular ideal 2-simplexes.

So, if $n = 2$, we could not conclude that $\tilde{f}_\infty$ agrees with g on the vertices of each simplex in the tesselation.

Theorem (W. Thurston). *Suppose M_1 and M_2 are orientable hyperbolic 3-manifolds and $f : M_1 \to M_2$ has non-zero degree d. If $\mathrm{vol}\, M_1 = |d| \,\mathrm{vol}\, M_2$ then f is homotopic to a covering map of degree d.*

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Thurston also defined a relative Gromov norm, which he used to show:

Theorem (W. Thurston). *If M_1 is a non-compact orientable hyperbolic 3-manifold of finite volume, and M_2 is obtained by Dehn-filling at least one cusp of M_1 then $\mathrm{vol}\, M_1 > \mathrm{vol}\, M_2$.*

Suppose Γ is a discrete subgroup of $\mathbb{H}^3$. For $x \in \mathbb{H}^3$ set
$\Gamma_x(\epsilon) = \{\gamma \in \mathsf{Isom}_+\mathbb{H}^3 \mid \mathrm{dist}(x, \gamma \cdot x) < \epsilon\}$

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Lemma (Special case of Margulis' lemma). *There exists a constant ϵ_0 with the following property:*

- *If $\Gamma < Isom_+{}^+\mathbb{H}^3$ is a discrete group and $x \in \mathbb{H}^3$ then $\langle \Gamma_x(\epsilon_0) \rangle$ is virtually nilpotent.*

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Lemma (Special case of Margulis' lemma). *There exists a constant ϵ_0 with the following property:*

- *If $\Gamma < Isom_+{}^+\mathbb{H}^3$ is a discrete group and $x \in \mathbb{H}^3$ then $\langle \Gamma_x(\epsilon_0) \rangle$ is virtually nilpotent.*

If Γ is torsion-free, i.e. if $\mathbb{H}^3/\Gamma$ is a manifold, the discrete, torsion-free, virtually nilpotent subgroups of Γ are actually abelian. There are three types:

- Cyclic groups generated by a loxodromic isometry;
- Cyclic groups generated by a parabolic isometry;
- Rank 2 free abelian groups generated by two parabolics.

The middle case can not arise if $\mathbb{H}^3/\Gamma$ has finite volume.

Definition. The *ϵ-thin part* $M_{(0,\epsilon]}$ of an orientable hyperbolic manifold M is the set of points $p \in M$ such that there is a geodesic loop of length $\leq \epsilon$ based at p. The ϵ-thick part is $M_{(\epsilon,\infty)} = M - M_{(0,\epsilon]}$

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Suppose $x \in M_{(0,\epsilon]}$, and let $\tilde{x}$ be a lift of x to $\mathbb{H}^3$. Then there exists $\gamma \in \Gamma$ such that $\mathrm{dist}(\tilde{x}, \gamma \cdot \tilde{x}) \leq \epsilon$.

For $G \subset \mathrm{Isom}_+\mathbb{H}^3$, define

$$C_\epsilon(G) = \{x \in \mathbb{H}^3 : \ \mathrm{dist}(\tilde{x}, g \cdot \tilde{x}) \leq \epsilon \text{ for some } g \in G\}.$$

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If $G \cong \mathbb{Z}$ is generated by a loxodromic isometry, then $C_\epsilon(G)$ is a banana (or empty). In this case $C_\epsilon(G)/G$ is a geometric tubular neighborhood of a geodesic.

If $G \cong \mathbb{Z}^2$ is generated by parabolic isometries, then $C_\epsilon(G)$ is a horoball and $C_\epsilon(G)/G$ is a cusp neighborhood.

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So, if M has finite volume and $\epsilon < \epsilon_0$ then $M_{(0,\epsilon]}$ is a union of cusp neighborhoods and tubes around short geodesics.

Theorem (Jørgensen). *For each $C > 0$ there exists a finite set $\{M_1, \ldots, M_k\}$ of finite-volume orientable hyperbolic 3-manifolds such that every orientable hyperbolic 3-manifold M with $\mathrm{vol}\, M < C$ is constructed by Dehn-filling some cusps of one of the M_i.*

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The idea is that there are only finitely many possible homeomorphism types for $M_{(\mu,\infty)}$ when $\operatorname{vol} M < C$. If $x_1, \ldots, x_n$ are points of $M_{(\mu,\infty)}$ with $\operatorname{dist}(x_i, x_j) > \mu$ then the balls $B(x_i\mu/2)$ are pairwise disjoint, so $n < C/v$ where $v = \operatorname{vol} B(x_i\mu/2)$. If $\{x_1, \ldots, x_n\}$ is maximal then every point of $M_{(\mu,\infty)}$ is within distance 2μ of some x_i. Thus there is a Delaunay "triangulation" of $\overline{M}_{(\mu,\infty)}$ with a bounded number of cells. (Lifted to $\mathbb{H}^3$, the 3-cells are convex hulls of sets of ≥ 4 points that lie on a sphere containing no lifts of x_i in its interior.

The proof of Thurston's hyperbolic Dehn-filling theorem implies:

Let M be an orientable finite-volume hyperbolic 3-manifold. Fix a set of cusps of M. For any $\epsilon > 0$, all but finitely many manifolds M' obtained by Dehn-filling these cusps have $|\operatorname{vol} M - \operatorname{vol} M'| < \epsilon$.

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Theorem. *The set of volumes of orientable hyperbolic 3-manifolds forms a well-ordered subset of $\mathbb{R}$, and there are only finitely many distinct manifolds with each volume.*

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Suppose $\operatorname{vol} M_1 > \operatorname{vol} M_2 > \cdots$. By passing to a subsequence we may assume each M_n is constructed by Dehn-filling of a given set of cusps of a manifold M. By Gromov's Theorem we have $\operatorname{vol} M > \operatorname{vol} M_n$ for all n. Thus $|\operatorname{vol} M - \operatorname{vol} M_n| > |\operatorname{vol} M - \operatorname{vol} M_1|$ for all $n > 1$. Contradiction.