Hyperbolic 3-manifolds

Marc Culler

GEAR Junior Retreat - July 2012

References:

W. Thurston, *The geometry and topology of* 3*-manifolds,*
`http://www.msri.org/publications/books/gt3m`

A. Hatcher, *Notes on basic* 3*-manifold topology,*
`http://www.math.cornell.edu/~hatcher/3M/3M.pdf`

W. Jaco and P. B. Shalen, "Seifert fibered spaces in 3-manifolds," *Mem. Amer. Math. Soc.* **21** No. 220 (1979).

G. P. Scott "The geometries of 3-manifolds," *Bull. London Math. Soc.* **15** 401-487 (1983).

Textbooks: Hempel, Benedetti and Petronio, Ratcliffe

The fine print

I will try to follow these guidelines, reserving the right to resort to hand-waving if I get stuck.

- Our 3-manifolds will be smooth. Usually they will be orientable. I will try to indicate when they may have boundary. They need not be compact, and will be said to be *closed* when they are compact without boundary.
- Surfaces in a 3-manifold M will be piecewise smooth, and properly embedded (i.e $\Sigma \cap \partial M = \partial\Sigma$) in the exceptional case where M has boundary
- Isotopies will be piecewise smooth.

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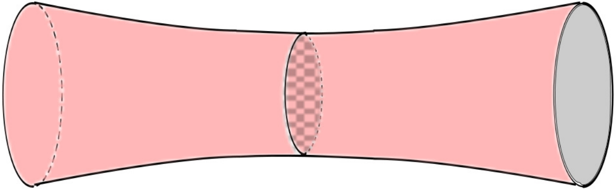

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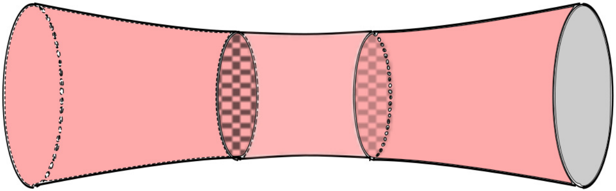

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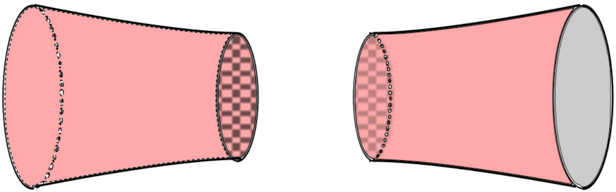

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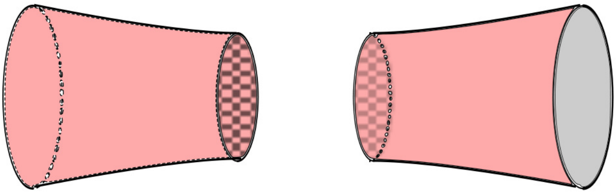

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When the simple closed curve ∂D is not the boundary of a disk in Σ, the surgery is called a *compression*. A compression never produces a sphere.

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But there are **lots** of irreducible 3-manifolds. Moreover, every closed 3-manifold has a unique description as a connected sum of prime 3-manifolds. (There may be $S^1 \times S^2$ summands, though, which are prime but not irreducible.)

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Two disjoint 2-spheres in a 3-manifold M are said to be *parallel* when they cobound $S^2 \times [0,1]$.

Kneser's Theorem. For any 3-manifold M there exists N_M such that if $\mathcal{S}$ is any family of 2-spheres which are pairwise disjoint and non-parallel then $|\mathcal{S}| \leq N_M$. (Hence M can have at most N_M non-trivial connected summands.)

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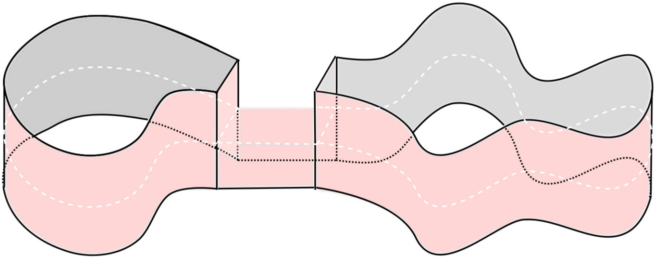

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- Reverse the sequence of surgeries to reconstruct the manifold bounded by Σ. At each stage, either some bounded region is expanded by attaching a 3-ball along a 2-disk, or reduced by removing a 3-ball attached along a 2-disk. Hence all of the bounded regions are balls at each stage.

Sketch of Kneser's Theorem

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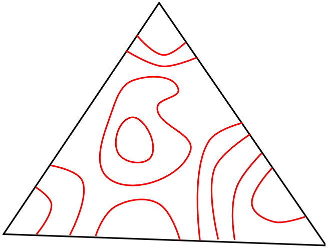

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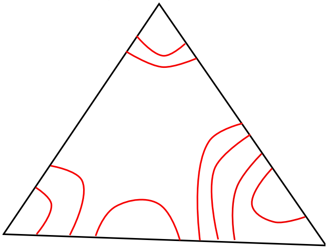

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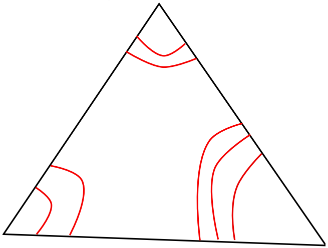

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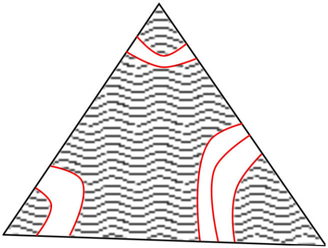

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Kneser's theorem gives existence of a prime decomposition. To prove uniqueness, first remove $S^1 \times S^2$ summands until no non-separating 2-spheres remain. Then show that any two maximal families of pairwise non-parallel spheres are isotopic.

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Theorem. *A compact irreducible 3-manifold M contains a family $\mathcal{T}$ of incompressible tori such that every connected 3-manifold obtained by cutting M along $\mathcal{T}$ is either Seifert-fibered or atoroidal. Up to isotopy there is a unique family $\mathcal{T}$ which is minimal under inclusion.*

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Among the eight geometries, the hyperbolic structures are generic. Non-hyperbolic geometric 3-manifolds are classified.

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By "analytic continuation" of ϕ, any smooth path in M starting at m lifts to a smooth path in X starting at x. Homotopic paths have homotopic lifts. If we fix a basepoint $\tilde{m}$ lying over m in the universal cover $\widetilde{M}$, then we obtain a unique *developing map* $D : (\widetilde{M}, \tilde{m}) \to (X, x)$ so that, for any path σ starting at m, if $\tilde{\sigma}$ is the lift of σ to $(\widetilde{M}, \tilde{m})$, then $D \circ \sigma$ is the lift of σ to (X, x).

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Theorem. *An X-structure defines a complete metric on M if and only if its developing map is a diffeomorphism $D : \widetilde{M} \to X$. In this case the holonomy representation is discrete.*

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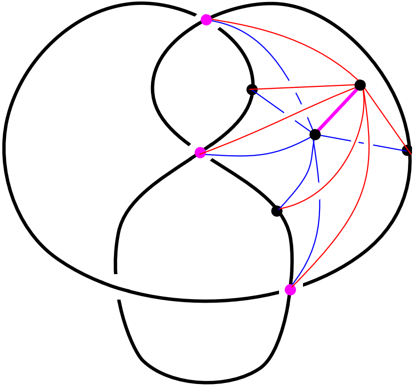

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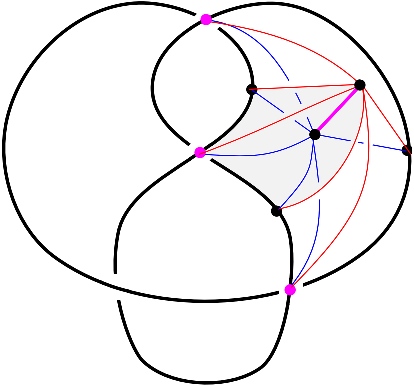

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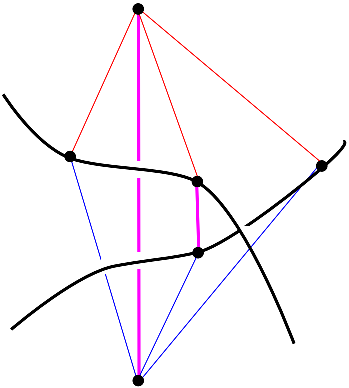

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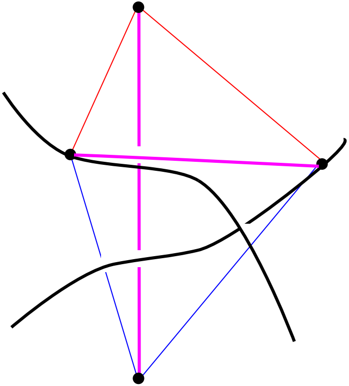

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To construct a complete hyperbolic structure on M it suffices to construct a diffeomorphism $D : \widetilde{M} \to \mathbb{H}^3$ carrying each ideal 3-simplex in $\widetilde{\mathcal{T}}$ to a geometric ideal simplex, i.e. the convex hull of 4 distinct points on S_∞. The map D will be the developing map of our hyperbolic structure.

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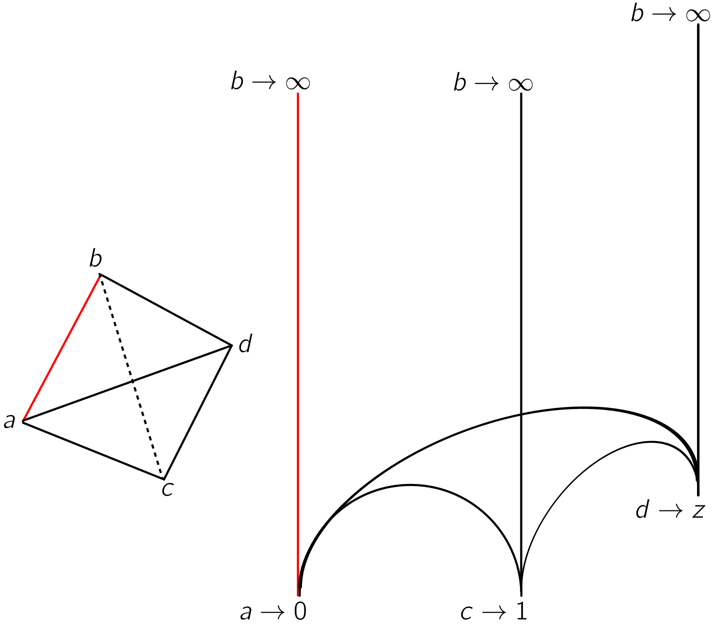
The first step is to solve the *gluing equations*. They have a variable z_i for each 3-simplex Δ_i in $\mathcal{T}$ and an equation for each edge. The value of z_i represents the cross-ratio (or shape parameter) of $D(\widetilde{\Delta}_i) \subset \mathbb{H}^3$ for any lift $\widetilde{\Delta}_i$ of Δ_i. (Fix an arbitrary ordering of the vertices.)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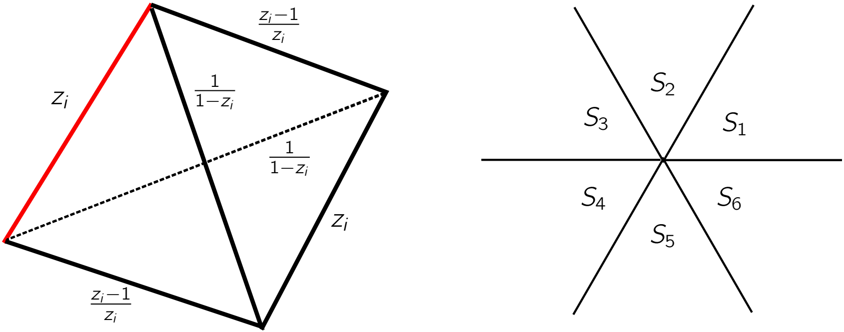

Let $S_1, \ldots, S_v$ be the linear fractions assigned to e in the tetrahedra incident to e. Then the equation corresponding to e is:

$$\prod_{i=1}^{v} S_i = 1.$$

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Given a solution, one can construct an equivariant map from $\widetilde{M}$ to $\mathbb{H}^3$, unique up to conjugation in $\mathrm{Isom}_+(\mathbb{H}^3)$, that carries topological ideal simplices to (possibly degenerate) geometric ideal simplices.

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- All $\operatorname{Im} z_i$ are non-zero with the same sign; and
- For each end, some (hence any) non-trivial curve on the torus has parabolic holonomy. (These are the *completeness equations*.)

Suppose that $Z = (z_1, \ldots, z_N)$ is a solution to the gluing equations which defines a complete hyperbolic structure; i.e all $\operatorname{Im} z_i > 0$, and the completeness equations hold.

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Consider a nearby solution $W = (w_1, \ldots, w_N)$, with all $\operatorname{Im} w_i > 0$, but not satisfying the completeness equations. The pseudo-developing map D defined by W is a developing map, but for an incomplete hyperbolic structure.

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For each end E of M, the holonomy representation ρ_D takes $\pi_1(E) \cong \mathbb{Z} \oplus \mathbb{Z}$ to an abelian group of loxodromic isometries with a common axis A_E. The metric space completion $\widehat{E}$ adjoins the quotient space $A_E/\rho_D(\pi_1(E))$ – either one point or a circle.

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In the case that $A_E/\rho_D(\pi_1(E))$ is a circle, $\widehat{E}$ is a hyperbolic manifold.

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Topologically, $\widehat{M}$ is obtained from N by adding a solid torus $S^1 \times D^2$ to the boundary component corresponding to E, so that the meridian curves $* \times \partial D$ are homotopic to μ_E. We say $\widehat{M}$ is a *Dehn filling* of N.

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Thurston's Dehn Filling Theorem. Let N be a compact 3-manifold boundary a torus. Then all but finitely many Dehn fillings of N are hyperbolic. (In fact there is a neighborhood of the developing map of M contains developing maps for hyperbolic structures on all but finitely many Dehn fillings.)

There is also an extension of this result to the case where ∂N has more than one boundary components.