GEAR Culler Problems

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1. Problem Set 1

1.1. **Problem 1.** Call the quotient 3-manifold M. In M, the front two triangles are glued together to form one triangle, with glueing instructions on its boundary. The glueing on the sides of this triangle make it an embedded Möbius strip in M, which we call S.

Consider the "flattening" map, $\pi : M \to S$, obtained by flattening the picture onto the front two triangles. For each point in $int(S)$ (that is, the closed triangle minus the double-arrowed side), there is an open neighborhood U around it so that $\pi^{-1}(U) \cong U \times [0,1]$. This makes M into an I-bundle over S. Since $\pi_1(S) \cong \mathbb{Z}$, there are only two possibilities for M. After checking that the triangles are glued with an orientation-reversing map, it is clear that M is orientable. Thus M is a twisted I-bundle over S, which is homeomorphic to a solid torus.

While there are many possible results of glueing together solid tori (any Lens space $L(p,q)$), "two of these" indicates that the glueings must respect the triangulation of the torus boundary given by the back two faces in the figure. Let x be the closed curve given by the single-arrowed edge, y the double-arrowed, and $x+y$ the unmarked edge in the triangulation of the torus boundary, while x', y', and $x'+y'$ will be these curves on another copy of M, which we call M'.

One can check the following are the only possibilities for simplicial maps of the boundary:

(1) $\{x, y, x+y\} \mapsto \{x', y', x'+y'\}$
(2) $\{x, y, x+y\} \mapsto \{y', x', x'+y'\}$
(3) $\{x, y, x+y\} \mapsto \{-y', x'+y', x'\}$
(4) $\{x, y, x+y\} \mapsto \{-x', x'+y', y'\}$
(5) $\{x, y, x+y\} \mapsto \{x'+y', -x', y'\}$
(6) $\{x, y, x+y\} \mapsto \{x'+y', -y', x'\}$

The homeomorphism type of ${}^{M \sqcup M'}/{\sim}$ is determined by where the meridians are glued. The meridians are given by $m = y - 2x$ and $m' = y' - 2x'$ (this curve is non-trivial on the torus boundary, but S provides a compressing disk in M), while the longitudes are $l = x$ and $l' = x'$.

One can check that in the possibilities listed above:

(1) $(m, l) \mapsto (m', l')$
(2) $(m, l) \mapsto (-2m' - 3l', m' + 2l')$
(3) $(m, l) \mapsto (3m' + 7l', -m' - 2l')$
(4) $(m, l) \mapsto (m' + l', -l')$
(5) $(m, l) \mapsto (-2m' - 7l', m' + 3l')$
(6) $(m, l) \mapsto (-3m' - 8l', m' + 3l')$

These give rise to

(1) $S^1 \times S^2$
(2) $L(2,3)$
(3) $L(3,7)$
(4) $L(1,1)$
(5) $L(2,7)$
(6) $L(3,8)$

1.2. **Problem 2.** A hyperbolic structure on a 3-manifold M gives rise to a discrete, faithful representation $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$. Let $\Gamma = \rho(\pi_1(M))$. Since M is closed, γ is loxodromic, for all $\gamma \in \Gamma$, since every curve has some definite translation length (making parabolics impossible). If M contains an embedded incompressible torus, there would exist a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of Γ, i.e. two commuting Möbius transformations $\alpha, \beta \in \Gamma$. Commuting loxodromic Möbius transformations share the same axis. If α and β have no common power, they act non-discretely on their shared axis, while if they have a common power ρ was not faithful.

If M is not closed, the previous argument shows that $\mathbb{Z} \oplus \mathbb{Z} < \Gamma$ must be a parabolic subgroup. Form the cover of M corresponding to this subgroup, and note that it is topologically $\mathbb{T}^2 \times \mathbb{R}$. Thus we may isotope the embedded torus out into the end. A careful solution would proceed to see that the torus is actually isotopic to $\mathbb{T}^2 \times \{1\}$, for which one must do a surgery argument similar to one in the proof of Alexander's Theorem.

1.3. **Problem 3.** An embedding of S^2 into a hyperbolic manifold lifts to an embedding into $\mathbb{H}^3$, since S^2 is simply-connected. The lift may be chosen so that it lies in a fundamental domain (for the action of the fundamental group by deck transformations). By Alexander's Theorem, this sphere bounds a ball, so the original S^2 bounded a ball in the manifold.

1.4. **Problem 4.** Recall that the trefoil knot is fibered: Its complement fibers over the circle, with fiber a punctured torus, and monodromy of order 6. In order to triangulate the complement, we will triangulate a cover equivariantly, then push down. The cover will be the one corresponding to the commutator subgroup of the fundamental group of the fiber. (This is a precise way of saying we take the cover of the punctured torus that looks like $\mathbb{R}^2 \setminus \mathbb{Z}^2$, and the universal cover of the base). Note that, in these coordinates, the (missing) knot has lifts winding through the (missing) $\mathbb{Z}_2$ points.

The monodromy map may be written as the order 6 matrix $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. The triangles with vertices $\{(0,0),(0,1),(1,0)\}$ and $\{(1,0),(0,1),(1,1)\}$ (and their $\mathbb{Z}^2$-translates), have images under the monodromy triangles with vertices $\{(0,0),(-1,0),(1,1)\}$ and $\{(-1,0),(0,1),(1,1)\}$ (and their $\mathbb{Z}^2$-translates). One may put two layers of tetrahedra down between these two layers of triangles, all of whose vertices are ideal, and so that the 'horizontal' faces coincide with the chosen triangles on the planes. Since this set of tetrahedra is preserved under the monodromy, and under the $\mathbb{Z}^2$-translations, it descends to an ideal triangulation, with 2 tetrahedra, of the trefoil knot complement. Attempting to solve Thurston's gluing equations will yield a family of degenerate real solutions: Ideal tetrahedra in $\mathbb{H}^3$ that collapse onto a plane.

1.5. **Problem 5.** The punctured torus may be given a hyperbolic structure invariant under the order 6 monodromy given above. This can be seen by taking a hexagonal torus, with order 6 symmetry, and deleting the fixed point. Now this hexagonal punctured torus can be given a complete hyperbolic structure via uniformization, so that the order 6 conformal symmetry is a hyperbolic isometry. This gives the $\mathbb{H}^2 \times \mathbb{R}$ structure. (One could also interpret a solution to the gluing equations for the triangulation given above as this $\mathbb{H}^2 \times \mathbb{R}$ structure, once checking completeness – there are many incomplete structures).

The $\widetilde{\mathrm{SL}_2 \mathbb{R}}$ structure is harder to see, but follows from the fact that the trefoil knot complement is homeomorphic to ${}^{\mathrm{SL}_2(\mathbb{R})}/_{\mathrm{SL}_2(\mathbb{Z})}$. This can be seen as follows (this argument is in Milnor, attributed to Quillen):

Note that the trefoil complement is homeomorphic to $S^3 \setminus \{w^2 + z^3 = 0\} \subset \mathbb{C}^2$. ${}^{\mathrm{SL}_2(\mathbb{R})}/_{\mathrm{SL}_2(\mathbb{Z})}$ identifies with ismorphism classes, up to homothety, of lattices in $\mathbb{C}$.

Each lattice in $\mathbb{C}$ is determined by its two Weierstrass invariants, the coefficients in the differential equation

$$(\wp')^2 = \wp^3 + g_2\wp + g_3$$

Scaling the lattice by t scales the invariants as $(g_2, g_3) \mapsto (t^{-4}g_2, t^{-6}g_3)$, so there is some homothety such that the invariants, as a point in $\mathbb{C}^2$, lie in S^3. Since the $\wp$-function gives a branched covering from a torus to $\mathbb{CP}^1$, branched over *distinct* points, the roots of the equation

$$Y^2 = X^3 + g_2X + g_3$$

are all distinct. Thus $4g_2^3 - 27g_3^2 \neq 0$. This may be normalized to give the map from ${}^{\mathrm{SL}_2(\mathbb{R})}/_{\mathrm{SL}_2(\mathbb{Z})}$ to the trefoil complement. Any pair (g_2, g_3) satisfying this inequality are obtained from a lattice.

2. Problem Set 2

2.1. **Problem 1.**

Lemma. For all $\gamma \in \mathrm{PSL}_2(\mathbb{C})$, $\epsilon > 0$, and sets $1 \neq A \subset \mathrm{PSL}_2(\mathbb{C})$,

$$\gamma \cdot C_\epsilon(A) = C_\epsilon(\gamma A \gamma^{-1})$$

Pf. $\exists\, 1 \neq g \in A$ and $x \in \mathbb{H}^3$ such that $d(x, g \cdot x) < \epsilon \iff$
$\exists\, 1 \neq g \in A$ and $y \in \mathbb{H}^3$ such that $d(y, \gamma g \gamma^{-1} \cdot y) < \epsilon$. $\square$

Suppose $G = \langle g \rangle$. If $\gamma \in \Gamma \setminus G$ and $x \in \gamma \cdot C_\epsilon(G) \cap C_\epsilon(G)$, then

$$\max\{d(\ x,\ g^n \cdot x\ ), d(\ x,\ \gamma g^m \gamma^{-1} \cdot x\ )\} < \epsilon$$

for some non-zero integers m and n. By the Margulis Lemma (and the choice of ϵ), g^n and $\gamma g^m \gamma^{-1}$ commute, and therefore have the same fixed points. Thus

$$\mathrm{Fix}(g) = \mathrm{Fix}(g^n) = \mathrm{Fix}(\gamma g^m \gamma^{-1}) = \gamma \cdot \mathrm{Fix}(g^m) = \gamma \cdot \mathrm{Fix}(g)$$

and $\mathrm{Fix}(g) \subset \mathrm{Fix}(\gamma)$.

If G is maximal cyclic loxodromic, then g is loxodromic, and γ has the same fixed points as g. If g and γ share a common power, then by maximality $\gamma \in G$. In this case, $\gamma \cdot C_\epsilon(G) = C_\epsilon(G)$. If they don't share a common power, they act non-discretely on their common axis, a contradiction.

If G is maximal parabolic, then g is parabolic. If γ is parabolic, then $\gamma \in G$ and again $\gamma \cdot C_\epsilon(G) = C_\epsilon(G)$. If γ is loxodromic, then its axis travels directly into the cusp, and cannot be the lift of a simple closed curve.

2.2. **Problem 2.** $\gamma(z) = (1+\delta)e^{\frac{2\pi i}{3}} z$, for δ small enough.

2.3. **Problem 3.** Let $\mathbb{F}_\infty = \langle x_n \rangle_{n=1}^{\infty}$. We form the HNN-extension with $\mathbb{F}_\infty$ as vertex and edge groups: Consider the injective homomorphisms

$$\begin{aligned} \iota_1 = id : \mathbb{F}_\infty &\to \mathbb{F}_\infty \\ \iota_2 : \mathbb{F}_\infty &\to \mathbb{F}_\infty \\ x_n &\mapsto x_{n+1} \end{aligned}$$

Then $\mathbb{F}_\infty *_{\mathbb{F}_\infty} \cong \langle x_n, t \mid t x_n t^{-1} = x_{n+1} \rangle_{n=1}^{\infty} \cong \langle x_1, t \rangle \cong \mathbb{F}_2$

2.4. **Problem 4.**

2.5. **Problem 5.** $B \in \mathrm{SL}_2(\mathbb{C})$ has characteristic polynomial given by $p_B(x) = x^2 - (\mathrm{tr}B)x + 1$. By the Cayley-Hamilton Theorem,

$$\begin{aligned} B^2 - (\mathrm{tr}B)B + I = 0 \quad &\iff \\ B + B^{-1} = (\mathrm{tr}B)I \quad &\iff \\ AB + AB^{-1} = (\mathrm{tr}B)A \quad &\iff \\ \mathrm{tr}AB + \mathrm{tr}AB^{-1} = (\mathrm{tr}A)(\mathrm{tr}B) \end{aligned}$$

2.6. **Problem 6.** We proceed by induction on the maximum number of times a generator g appears in a word γ:

If each generator appears in γ no more than once, the conclusion is immediate. Suppose now, WLOG, $\gamma = v_1 g v_2 g^\epsilon v_3$, for some (possibly identity) words v_1, v_2, v_3, $g = g_i$ for some i, and $\epsilon \in \{1, -1\}$. We use Problem 5 repeatedly, and the fact that trace is invariant under conjugation:

$$\mathrm{tr}(v_1 g v_2 g^\epsilon v_3) + \mathrm{tr}(v_1 g v_2 v_3^{-1} g^{-\epsilon}) = \mathrm{tr}(v_1 g v_2)\mathrm{tr}(g^\epsilon v_3)$$
$$\mathrm{tr}(v_1 g v_2 v_3^{-1} g^{-\epsilon}) + \mathrm{tr}(g^{-1} v_1^{-1} v_2 v_3^{-1} g^{-\epsilon}) = \mathrm{tr}(v_1 g)\mathrm{tr}(v_2 v_3^{-1} g^{-\epsilon})$$

Case 1. For $\epsilon = 1$,

$$\mathrm{tr}(g^{-1} v_1^{-1} v_2 v_3^{-1} g^{-1}) + \mathrm{tr}(g v_1^{-1} v_2 v_3^{-1} g^{-1}) = \mathrm{tr}(g)\mathrm{tr}(v_1^{-1} v_2 v_3^{-1} g^{-1})$$

Collecting terms,

$$\mathrm{tr}(\gamma) = \mathrm{tr}(v_1 g v_2)\mathrm{tr}(g v_3) - \mathrm{tr}(v_1 g)\mathrm{tr}(v_2 v_3^{-1} g^{-1}) + \mathrm{tr}(g)\mathrm{tr}(v_1^{-1} v_2 v_3^{-1} g^{-1}) - \mathrm{tr}(v_1^{-1} v_2 v_3^{-1})$$

Case 2. For $\epsilon = -1$,

$$\mathrm{tr}(\gamma) = \mathrm{tr}(v_1 g v_2)\mathrm{tr}(g^{-1} v_3) - \mathrm{tr}(v_1 g)\mathrm{tr}(v_2 v_3^{-1} g) + \mathrm{tr}(v_1^{-1} v_2 v_3^{-1})$$

In either case, we have expressed $\mathrm{tr}(\gamma)$ as a polynomial in the traces of words in which the maximum number of occurrences of a generator has been reduced.

2.7. **Problem 7.** Choose any ideal triangulation for the fundamental domain for M. (Note: This is straightforward only because we have assumed that M admits a fundamental domain with only ideal vertices. In general this existence is unknown). These tetrahedra determine triangulations of the polyhedral fundamental domain for M, but the face identifications may not respect these face triangulations. For every pair of triangles on one face which have a common edge (determining four vertices), one may 'switch diagonals', replacing these two triangles with two new ones. This move is achieved by pasting on a flat ideal tetrahedron which is the convex hull of the four vertices. Interpret triangulations of a polygon as the vertices in a graph, whose edges correspond to the 'switch diagonals' move. Since switches can be applied to reach a fixed canonical position, this graph is connected, and the 'flat triangulation' of a hyperbolic manifold can be achieved.