Combinatorial Teichmüller spaces.

1. Consider the set **Hex** of hyperbolic right-angled hexagons (with a base vertex) up to orientation preserving isometry. Prove that the lengths of any three alternating sides defines a bijection

$$\mathbf{Hex} \to \mathbb{R}^3_+.$$

2. Fact: Given a pants decomposition of a surface, and a curve γ, the length of γ is a convex function of any twist coordinate. Moreover, if γ has nonzero intersection number with the i^{th} curve, then it is a strictly convex function of the i^{th} twist parameter.

Using this, construct a set of $9g-9$ curves $A = \{\alpha_1, \dots, \alpha_{9g-9}\}$ on a closed surface S_g of genus g with the property that the lengths of the curves in A uniquely determines any point in the Teichmüller space of S_g. That is, the map $\mathcal{T}(S_g) \to \mathbb{R}^{9g-9}_+$ defined by

$$X \mapsto (\ell_{\alpha_1}(X), \dots, \ell_{\alpha_{9g-9}}(X))$$

is injective.

3. Given two curves α, β on a hyperbolic surface S (not equal to the punctured torus or four-punctured sphere), prove that the curve complex distance between α and β is bounded in terms of intersection numbers by

$$d_C(\alpha, \beta) \leq 2i(\alpha, \beta) + 1.$$

4. Prove that the Farey graph is δ–hyperbolic. Hint: use the embedding into the Teichmüller space of a (punctured) torus = the hyperbolic plane.

5. Prove that for any closed surface S_g of genus g, there is a number c_g so that for any hyperbolic structure X on S_g, the length of the shortest essential closed geodesic with respect to X is at most c_g. Prove that there is a constant b_g so that for any hyperbolic structure X on S_g, there exists a pants decomposition $\mathcal{P}$ so that the lengths of all the curves in $\mathcal{P}$ with respect to X are at most b_g. Show that for any $\epsilon > 0$, the ϵ–thick part of moduli space is compact.

6. Prove that the curve complex of S has infinite diameter. ...to be discussed...

7. Consider the embedding of the Farey graph into the Teichmüller space of a punctured torus (= hyperbolic plane) for which the edges are hyperbolic geodesics. Prove that the edges are Weil-Petersson geodesics. Show that Weil-Petersson geodesics exiting every compact subset of this Teichmüller space are uniquely determined by their limit point on $\mathbb{R} \cup \{\infty\}$.

8. Let X be the punctured torus. Using the hints from lecture, show that the map Q that associates to each simple closed curve the corresponding maximally noded surface is a quasi-isometry from the Farey graph to the Weil-Petersson metric on $\mathrm{Teich}(X)$.