SOLUTIONS OF SOME EXERCISES ON DIVISIBLE CONVEX SETS

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Here come short solutions for some exercises of the exercises sheets A, B, C, D of Yves Benoist. I might expand them later on.
We call $p : \mathbb{R}^d \longrightarrow \mathcal{P}(\mathbb{R}^d)$ the canonical projection.

Sheet A.

Exercise 1. The symmetry of d_Ω and its definiteness are clear. To prove the triangular inequality, first remark that if $x, z, y \in \Omega$ are in a same projective line, then we have

$$d_\Omega(x, y) = d_\Omega(x, z) + d_\Omega(z, y).$$

(That means that projective lines are actually metric geodesics.)
Now, pick two points x and y, and another point z which is not on (xy). We use Figure 1. The points a_1 and b_1 are the intersection points of (xy) and the boundary. The point o is the intersection point of two supporting hyperplanes of Ω at a_1 and b_1. The point z' is the intersection of (oz) with (xy). We are going to prove that

$$d_\Omega(y, z') \leqslant d_\Omega(y, z) \text{ and } d_\Omega(z', x) \leqslant d_\Omega(z, x),$$

which would imply that $d_\Omega(x, y) = d_\Omega(x, z') + d_\Omega(z', y) \leqslant d_\Omega(x, z) + d_\Omega(z, y)$ by the previous remark.
Recall that

$$d_\Omega(x, y) = \log[a_1 b_1 x y] = \frac{|a_1 x|}{|a_1 y|}\frac{|b_1 y|}{|b_1 x|}.$$

($|\cdot|$ is any auxiliary norm on the affine line (xy).) We use the fundamental property of the cross-ratio which says that

$$[a_1 b_1 z' y] = [a_2' b_2' z y] \text{ and } [a_1 b_1 x z'] = [a_3' b_3' x z].$$

Now, just notice that

$$[a_2' b_2' z y] \leqslant [a_2 b_2 z y] \text{ and } [a_3' b_3' x z] \leqslant [a_3 b_3 x z], \tag{1}$$

hence

$$[a_1 b_1 z' y] \leqslant [a_2 b_2 z y] \text{ and } [a_1 b_1 x z'] \leqslant [a_3 b_3 x z],$$

and taking logarithm :

$$d_\Omega(y, z') \leqslant d_\Omega(y, z) \text{ and } d_\Omega(z', x) \leqslant d_\Omega(z, x).$$

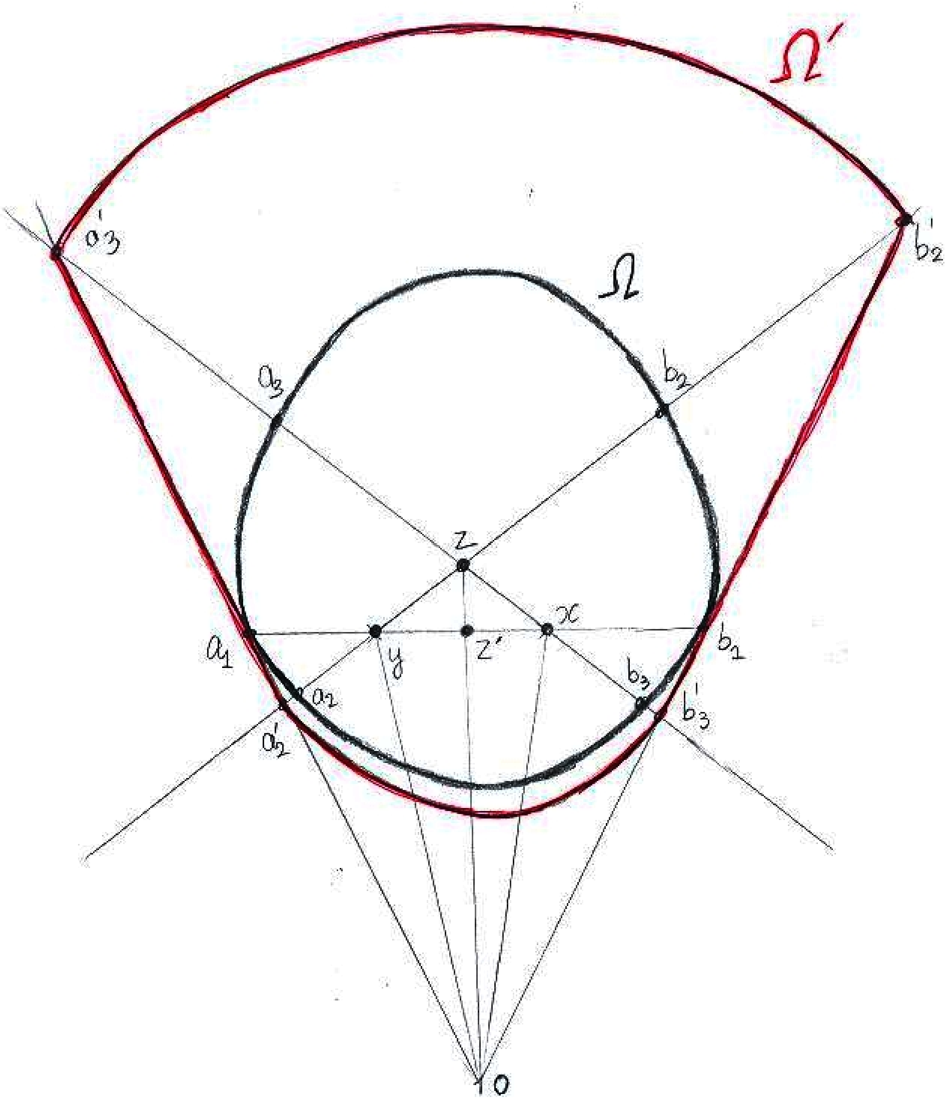

FIGURE 1

Furthermore, we can see that there is equality in (1) if and only if $a_2 = a_2'$, $b_2 = b_2'$, $a_3 = a_3'$ and $b_3 = b_3'$. So, if one considers the red convex set Ω' of the picture, we will have

$$d_\Omega(x, y) = d_\Omega(x, z) + d_\Omega(z, y),$$

and the space will not be uniquely geodesic. In fact, the only obstruction to be uniquely geodesic is to have two open segments in the boundary $\partial\Omega$ which are in a same 2-dimensional projective subspace but which are not included in a same supporting hyperplane.
In particular, (Ω, d_Ω) is uniquely geodesic if Ω is strictly convex.

The metric d_Ω is complete because metric balls are compact.

Exercise 2.

(i) One has to prove that, if K is a compact subset of Ω, then the set $\{g \in \Delta,\ gK \cap K \neq \emptyset\}$ is finite. This is a consequence of Arzelá-Ascoli theorem and the fact that the elements of Δ preserve the metric d_Ω :

Theorem 1 (Arzelá-Ascoli)**.** *Let X be a compact Hausdorff space, and (Y, d_Y) a metric space. A subset $F \subset \mathcal{C}(X,Y)$ of continuous functions from X to Y is relatively compact if and only if F is*

(i) equicontinous, that is, for all $\epsilon > 0$ and $x \in X$, there is a neighborhood U of x in X such that for all $y \in U$ and $f \in F$, one has $d_Y(f(x), f(y)) < \epsilon$;

(ii) pointwise relatively compact, that is, the set $\{f(x),\ f \in F\}$ is relatively compact for any $x \in X$.

Use the theorem with $X = K$ and $(Y, d_Y) = (\Omega, d_\Omega)$.

(ii) Let Ω_0 be the convex hull of the Δ-orbit of $x_0 \in \Omega$.
A convex set is the convex hull of its extremal points. So, if $\Omega_0 \neq \Omega$ then there is an extremal point z of Ω which is not in the closure $\overline{\Omega}_0$ of Ω_0. This point has a neighborhood U in the projective space $\mathcal{P}(\mathbb{R}^d)$ such that $U \cap \Omega_0 = \emptyset$.
Take a sequence (z_n) in $U \cap \Omega_0$ converging to z. Then one can check that the distance $d_\Omega(z_n, \Omega_0)$ goes to $+\infty$ with n. In particular the distance $d_\Omega(z_n, \Delta \cdot x_0)$ goes to $+\infty$, which contradicts the fact that the action of Δ on Ω is cocompact.

Sheet B.

Exercise 1. If $\pi = \lim \frac{\gamma_n}{\|\gamma_n\|}$ is a rank-one operator, call $\pi^+ = p(\pi(\mathbb{R}^d))$ its image in $\mathcal{P}(\mathbb{R}^d)$), which is a point. Set Λ_Γ as the closure of the points π^+ for any limit $\pi = \lim \frac{\gamma_n}{\|\gamma_n\|}$, $\gamma_n \in \Gamma$, which is a rank-one operator. This is the smallest Γ-invariant closed subset of $\mathcal{P}(\mathbb{R}^d)$.

Exercise 5.

(i) The boundary $p(\partial C)$ of $p(C)$ is Γ-invariant and closed, hence contains Λ_Γ.

(ii) The convex hull $\Omega_{min} = C(\Lambda_\Gamma)$ of Λ_Γ in $\mathcal{P}(\mathbb{R}^d)$ is contained in any other Γ-invariant convex subset of $\mathcal{P}(\mathbb{R}^d)$. Take $C_{\min}$ as one of the two connected components of $C_{min} = p^{-1}(\Omega_{min})$. For C_{max}, consider the dual action of Γ on the dual space $(\mathbb{R}^d)^*$. It preserves a unique minimal convex cone C^*_{max}. By duality, C^*_{max} is sent on a properly convex cone C_{max} of $\mathbb{R}^d$. Since duality reverses inclusions, C_{max} contains any Γ-invariant convex cone.

(iii) This is a consequence of Ex.2 (ii) of sheet A.

Sheet C.

Exercise 1.

(i) The non-trivial algebraic subgroups of $\mathrm{SL}(2,\mathbb{R})$ are, up to conjugation, the groups of diagonal matrices, of upper-triangular matrices, of unipotent matrices and $\mathrm{SO}(2,\mathbb{R})$. All of them are solvable, so Γ is Zariski-dense in $\mathrm{SL}(2,\mathbb{R})$.

(ii) Let $\overline{\Gamma}$ be the topological closure of Γ in $\mathrm{SL}(2,\mathbb{R})$. It is a Lie subgroup of $\mathrm{SL}(2,\mathbb{R})$ whose Lie algebra is invariant under the adjoint action of $\mathrm{SL}(2,\mathbb{R})$ since Γ is Zariski-dense. Hence its Lie algebra is either $\{0\}$ or $\mathfrak{sl}(2,\mathbb{R})$, that is, Γ is discrete or $\overline{\Gamma} = \mathrm{SL}(2,\mathbb{R})$ hence Γ is dense.

(iii) (One should read the question as : find a matrix whose eigenvalues are real, positive and distinct from 1.) If Γ is dense, it is obvious. If Γ is discrete, one can look at its action on the hyperbolic plane $\mathbb{H}^2$. One wants to prove that Γ contains a hyperbolic transformation. If not, either Γ contains two parabolic transformations fixing different points of $\partial\mathbb{H}^2$ and by considering their product one can construct a hyperbolic element (maybe one should consider powers of them) ; or Γ contains two elliptics elements (rotations) fixing two distinct points in $\mathbb{H}^2$ and we can do the same (that is a bit trickier).

(iv) Consider two elements whose eigenvalues are real and positive and follow the arrows to construct one whose eigenvalues are real and negative...

Exercise 4. Let N be an infinite abelian normal subgroup of Γ. $\mathrm{SL(d},\mathbb{R})$ is the Zariski-closure of Γ in $\mathrm{SL(d},\mathbb{R})$. Call H the Zariski-closure of N in $\mathrm{SL(d},\mathbb{R})$. H is abelian and normal in $\mathrm{SL(d},\mathbb{R})$, because these notions involve only polynomial relations which are preserved when one goes to the Zariski-closure. But $\mathrm{SL(d},\mathbb{R})$ is a simple group, hence H should be $\{Id\}$ or $\mathrm{SL(d},\mathbb{R})$, which contradicts the fact that is respectively infinite and abelian.

Sheet D.

Exercise 4.

(i) Call s an open segment in $\partial\Omega$. One can find a triangle T with vertices x, y, z which contains Ω and such that $s \subset [xy]$. Now consider the projective transformation γ given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

in the basis x, y, z. γ preserves the triangle and $\gamma^n(\Omega)$ converges to a triangle when n goes to $+\infty$ (the vertices of this triangle are z and the extremal points of the maximal closed segment of $\partial\Omega$ containing s).

(ii) One can proceed by duality using part (i). One can also consider a triangle containing Ω whose one of the vertices is a point $x \in \partial\Omega$ where $\partial\Omega$ is not $\mathcal{C}^1$.

Exercise 5.

(i) Fix an affine chart containing Ω and choose a Euclidean metric on it. There is a point $x \in \partial\Omega$ where the Hessian matrix of the boundary is non-degenerate. Let B be the osculating ball of $\partial\Omega$ at x. The Hilbert geometry (B, d_B) is a hyperbolic geometry. Consider a hyperbolic transformation γ of B whose repulsive fixed point is x and attractive fixed point is some $y \in \partial B$. Because the boundaries of Ω and B are the same at x up to order 2, one can check that $\lim\limits_{n\to+\infty} \gamma^n(\Omega) = B$.

(ii) If Ω is a polygon, $\overline{G_d\Omega}$ consists of the polygone and a triangle. If Ω is a quarter disk, $\overline{G_d\Omega}$ consists of the quarter disk, a triangle, a half disk and an ellipsoid.

Exercise 6.

(i) Assume the contrary. One can find a sequence of strictly convex sets Ω_n of F and a sequence of geodesic triangles $T_n = x_n y_n z_n$ in Ω_n as well as a point u_n on $[x_n y_n]$ such that

$$\min\{d_{\Omega_n}(u_n, [x_n z_n]), d_{\Omega_n}(u_n, [y_n z_n])\} \geqslant n.$$

Benzécri's compactness theorem says that the action of G_d on the space $\{(\Omega, x), \Omega \subset \mathcal{P}(\mathbb{R}^d),\ x \in \Omega\}$ is proper and cocompact. Since F is closed, one can assume that (Ω_n, u_n) converges to (Ω, u), with Ω stricly convex. By passing to a subsequence, one can also assume that x_n, y_n and z_n converge to x, y and z in $\overline{\Omega}$. But Ω is strictly convex, so $d_\Omega(u, [xz]) < +\infty$, which contradicts the fact that

$$\min\{d_{\Omega_n}(u_n, [x_n z_n]), d_{\Omega_n}(u_n, [y_n z_n])\} \to +\infty.$$

(ii) Take a limit $\Omega = \lim \Omega_n$ with Ω_n in F_δ. Pick a triangle T in Ω. It is a limit of triangles in Ω_n which are δ-thin, so T is also δ-thin.

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