TRANSLATION SURFACES, INTERVAL EXCHANGE MAPS, AND COUNTING PROBLEMS: GEAR PROBLEM SESSIONS

JAYADEV S. ATHREYA

1. Interval Exchange Maps and Translation Surfaces.

(1) Prove that for a translation surface of genus g, the sum of the excess angle at singular points must be $(4g-4)\pi$. Equivalently, show that for any holomorphic 1-form on a genus g Riemann surface, the orders of the zeros must sum to $2g-2$.

(2) Show that the regular $4g$-gon with parallel sides identified yields a genus g surface with one singular point of angle $(4g-2)\pi$. Show that the regular $4g+2$-gon with opposite sides identified yields a genus g surface with two singular points each of angle $2g\pi$.

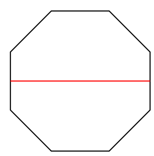
(3) Let $\alpha \in [0,1)$. Show that the circle rotation $R_\alpha : [0,1) \to [0,1)$ given by

$$R_\alpha(x) = x + \alpha \bmod 1$$

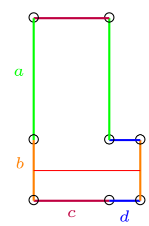
can be realized as an interval exchange of two intervals.

(4) Describe the interval exchange map associated to the flow with slope $\frac{1}{s}$, $s > 0$, to the horizontal transversals illustrated below (express your answer in terms of s)

(a) The regular octagon, assuming the octagon has height 1.



(b) An L-shaped table with parameters $a, b, c, d > 0$, with $a + \frac{b}{2} = 1$. Do it for $s < c$. Perhaps pick some specific values of a, b, c, d, s to make the problem more concrete (try $b = c = 1$, and let d vary as a free parameter to start).



(5) Prove that if $\alpha \notin \mathbb{Q}$, the orbit of any point $x \in [0,1)$ under R_α is dense in $[0,1)$. Move on to showing that it is in fact uniformly distributed with respect to Lebesgue measure.

(6) More generally, prove that a continuous transformation T of a compact metric space X with a unique invariant probability measure μ satisfies that for any $f \in C(X)$,

$$\frac{1}{N}\sum_{i=0}^{n-1} f(T^i x) = \int_X f\,d\mu$$

(7) Prove that an irreducible IET with length parameters $(\lambda_1, \ldots, \lambda_m)$ satisfies Keane's infinite distinct orbit condition (i.d.o.c.) (that is, the *negative* orbits of the discontinuity points $\delta_j = \sum_{i=1}^{j} \lambda_i$ are infinite and distinct) if $\sum \lambda_i = 1$ is the only rational relation satisfied by the λ_i. Show also that the i.d.o.c. implies that every well-defined forward orbit is dense.

(8) Prove that the set of holonomy vectors of saddle connections is a discrete subset of the plane.

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2. **Rauzy Induction, Lyapunov Exponents, and Teichmüller Flow.**
   (1) Show that Rauzy induction on the space of 2-IETS is the *Farey map* $f : [0,1] \to [0,1]$

   $$f(x) = \begin{cases} \frac{x}{1-x} & x \in [0,1/2) \\ \frac{1-x}{x} & x \in (1/2,1] \end{cases}$$

   Show further that if you define the 'speed-up' by applying each transformation as many times as you can, you obtain the *Gauss map* $G(x) = \left\{\frac{1}{x}\right\}$.

   (2) Compute the Rauzy graph for the permutation (4321).

   (3) $\star$ Take the transversal for the regular octagon, sheared by h_s for your favorite value of s from Problem 2(a), §1.1, and shorten it from the right until it hits another vertical singular leaf. Compute the new associated IET. How does it relate to the original IET? Do the same exercise for the L-shaped table in Problem (2)(b).

   (4) $\star$ Assuming Kingman's Subadditive Ergodic Theorem, prove (almost sure) existence of Lyapunov exponents for random products (matrices chosen by fair coin flip) of the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

   (5) Compute the Lyapunov exponents of a hyperbolic toral automoprhism, that is, the map $T_A : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ given by

   $$T_A(\mathbf{v}) = A\mathbf{v} \bmod 1,$$

   where $A \in SL(2,\mathbb{Z})$ has trace > 2. For example, let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

   (6) For your favorite IET $T_{\lambda,\pi}$, and point $x \in [0,1]$, use a computer program to compute the number of visits of the orbit of x to the jth interval up till time N. Subtract $n\lambda_j$, take the logarithm of what's left, and divide by $\log n$. Does it appear you are getting a limit?

   (7) Write a script to take random products of the matrices in Problem 1, §1.3. Apply them to a fixed (non-zero) vector $\mathbf{v} \in \mathbb{R}^2$. How quickly do you 'see' the Lyapunov exponents?

3. **Counting.**

3.1. *Integer Vectors and Lattices.*

(1) Prove that

(a) $\lim_{R\to\infty} \frac{|\mathbb{Z}^2 \cap B(0,R)|}{\pi R^2} = 1.$

(b) $\lim_{R\to\infty} \frac{|\mathbb{Z}^2_{prim} \cap B(0,R)|}{\pi R^2} = \frac{1}{\zeta(2)}.$

Also, compute the hyperbolic volume of $\mathbb{H}^2/SL(2,\mathbb{Z})$.

(2) Given $f \in C_c(\mathbb{R}^2)$, define $\widehat{f} : SL(2,\mathbb{R})/SL(2,\mathbb{Z}) \to \mathbb{R}$ by

$$\widehat{f}(gSL(2,\mathbb{Z})) = \sum_{\mathbf{v}\in\mathbb{Z}^2} f(g\mathbf{v}).$$

Show that

$$\int_{SL(2,\mathbb{R})/SL(2,\mathbb{Z})} \widehat{f} d\mu = \int_{\mathbb{R}^2} f dm,$$

where $d\mu$ and dm are, respectively the Haar probability measure on $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ and the standard Lebesgue measure on $\mathbb{R}^2$.

3.2. *The Siegel-Veech Machine.*

(1) Let $\mathcal{H}$ be a stratum of Ω_g, and let $\mu_{\mathcal{H}}$ denote the Masur-Veech probability measure on $\mathcal{H}$. Given $f \in C_c(\mathbb{R}^2)$, let $\widehat{f} : \mathcal{H} \to \mathbb{R}$ be given by

$$\widehat{f}(\omega) = \sum_{\mathbf{v}\in\Lambda_\omega} f(\mathbf{v}).$$

Assuming $\widehat{f} \in L^1(\mathcal{H}, \mu_{\mathcal{H}})$, show that there is a constant $c_{\mathcal{H}}$ so that

$$\int_{\mathcal{H}} \widehat{f} d\mu_{\mathcal{H}} = c_{\mathcal{H}} \int_{\mathbb{R}^2} f dm.$$

(2) Let Q be the quadrilateral with vertices at $(\pm 1, 1), (\pm 1/2, 1/2)$. Show that for $t >> 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} \chi_Q(g_t r_\theta \mathbf{v}) \approx \begin{cases} e^{-t} & |\mathbf{v}| \in (e^{t/2}/2, e^{t/2}) \\ 0 & \text{otherwise} \end{cases}$$

where r_θ is the rotation $r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, and $g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$

3.3. *Computer Experiments.*

(1) Write SAGE code to generate (primitive) lattice vectors and compute asymptotics.

(2) Write SAGE code to generate saddle connections on an L-shaped table and compute asymptotics. Try the parameter values $a = d = \frac{\sqrt{5}-1}{2}$ and $b = c = 1$.