Geometry, Groups and Dynamics/GEAR Seminar
(held at the Illinois hub of GEAR)

12:00 pm, Tuesday, April 5, 2016, 243 Altgeld Hall

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Homology and volume for hyperbolic 3-orbifolds, and enumeration of arithmetic groups

Abstract: A theorem of Borel's asserts that for any positive real number V, there are at most finitely many arithmetic lattices in ${\rm PSL}_2({\mathbb C})$ of covolume at most V, or equivalently at most finitely many arithmetic hyperbolic 3-orbifolds of volume at most V. Determining all of these for a given V is algorithmically possible for a given V thanks to work by Chinburg and Friedman, but appears to be impractical except for very small values of V, say $V=0.41$. (The smallest covolume of a hyperbolic 3-orbifold is about 0.39.) It turns out that the difficulty in the computation for a larger value of V can be dealt with if one can find a good bound on $\dim H_1(O,\mathbb Z/2 \mathbb Z)$, where O is a hyperbolic 3-orbifold of volume at most V. In the case of a hyperbolic 3-manifold M, not necessarily arithmetic, joint work of mine with Marc Culler and others gives good bounds on the dimension of $H_1(M,\mathbb Z/2 \mathbb Z)$ in the presence of a suitable bound on the volume of M. In this talk I will discuss some analogous results for hyperbolic 3-orbifolds, and the prospects for applying results of this kind to the enumeration of arithmetic lattices. A feature of the work that I find intriguing is that while it builds on my geometric work with Culler, the new ingredients involve primarily purely topological arguments about manifolds---the underlying spaces of the orbifolds in question---and have a classical, combinatorial flavor. At this point it appears that I can prove the following statement: If Ω is a hyperbolic 3-orbifold of volume at most 1.72, having a link as singular set and containing no embedded turnovers, then $$\dim H_1(\Omega;\mathbb Z_2)\le 1+ \max\bigg(3,7\bigg\lfloor\frac{10}{3}{\rm vol}(\Omega)\bigg\rfloor\bigg)+ \max\bigg(3, 7\bigg\lfloor\frac{5}{3}{\rm vol}(\Omega)\bigg\rfloor\bigg). $$ In particular, $\dim H_1(\Omega;\mathbb Z_2)\le50$. Various stronger bounds on $\dim H_1(\Omega;\mathbb Z_2)$ follow from stronger bounds on the volume of Ω. The restriction on turnovers is not an obstruction to applying the results to the enumeration of arithmetic groups. The assumption that the singular set is a link is more serious, but as it is used only in a mild way in this work, the methods seem promising for the prospective application.

Video