Junior GEAR Retreat
Complex projective structures and their holonomy limits

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1 Problem Session

These problems are related to the material from the mini-course lectures. Several of them focus on the properties of the Schwarzian derivative and generalizations thereof.

The more challenging or open-ended problems are marked with an asterisk ($*$).

1. (a) For what values of c is the quadratic differential $c\, dz^2 \in Q(\Delta)$ the Schwarzian derivative of a univalent map $\Delta \to \mathbb{CP}^1$?

   (b) For what values of c is the quadratic differential $\dfrac{c\, dz^2}{z^2} \in Q(\mathbb{H})$ the Schwarzian derivative of a univalent map $\mathbb{H} \to \mathbb{CP}^1$?

   (c) Generalize (a) and (b) to decompose the c-plane into sets where the corresponding map is k-to-one, for each $k \in \mathbb{N}$.

2. Let $\Gamma \subset \mathbb{C}$ denote the preimage of $\{|\mathrm{Re}(z)| = 1\}$ under the map $z \mapsto z^{3/2}$. Thus Γ consists of three bi-infinite paths (as in Figure 1) which we denote by $\gamma_1, \gamma_2, \gamma_3$. Orient each γ_i so that the origin lies to the left.

   Show that for each $i \in \{1, 2, 3\}$ the expression

   $$f_i(z) = \int_{\gamma_i} \exp(zu + \frac{1}{3}u^3)\, du$$

   defines an entire function and that the Schwarzian derivative of any ratio f_i/f_j is equal to $2zdz^2$.

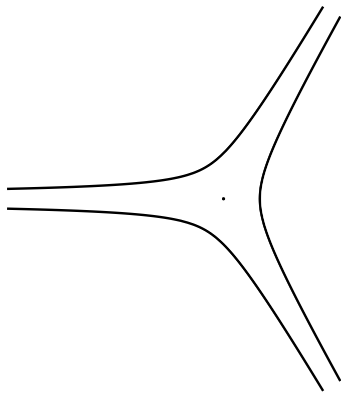

Figure 1: Contours of integration in Problem 2.

3. (*) Can the method of the previous problem be generalized to give a similarly explicit expression for a function whose Schwarzian derivative is a constant multiple of $z^n dz^2$, for $n > 1$?

4. There are two natural ways to construct a projective structure on a torus:

   - The quotient of $\mathbb{C}$ by a lattice acting by translations
   - The quotient of $\mathbb{C}^*$ by the cyclic group generated by a hyperbolic Möbius transformation $z \mapsto \lambda z$.

   If the lattice and cyclic group are chosen correctly (how?) then the two constructions may give isomorphic Riemann surfaces. In that case, determine the holomorphic quadratic differential representing the difference of the projective structures.

5. (a) Show that the Schwarzian tensor of a conformal metric is holomorphic if and only if the curvature of the metric is locally constant.

   (b) Conclude that a conformal metric with holomorphic Schwarzian tensor is always the pullback by a holomorphic map of one of the following "standard" conformal metrics:

   - A positive multiple of the spherical metric of $S^2 \simeq \mathbb{CP}^1$,
   - The Euclidean metric of $\mathbb{C}$, or
   - A positive multiple of the hyperbolic metric the unit disk $\Delta \subset \mathbb{C}$.

6. Show that the unit tangent bundle of hyperbolic space $\mathbb{H}^3$ can be naturally identified with the space of pairs $(p, \sigma^{(1)})$ where $p \in \mathbb{CP}^1$ and where $\sigma^{(1)}$ is the 1-jet of a conformal metric on a neighborhood of p. (Consider the way

that $\mathrm{PSL}_2\mathbb{C}$ acts on both spaces.) How is this related to the Epstein surface construction?

7. Explicitly determine the osculation map $\mathbb{C} \to \mathrm{PSL}_2\mathbb{C}$ of the entire function $\exp(z)$. Verify that its Darboux derivative gives the expected connection to the Schwarzian derivative.

8. (*) The mini-course discussed limits of holonomy representations for projective structures on a compact Riemann surface of genus $g > 1$. How much of this picture can be developed analogously for projective structures on an elliptic curve (i.e. a Riemann surface structure on the torus) and the limits of their holonomy representations $\mathbb{Z} \oplus \mathbb{Z} \to \mathrm{PSL}_2\mathbb{C}$?

9. (*) For $t \in \mathbb{R}^+$ let $\rho_t : \mathbb{F}_2 \to \mathrm{PSL}_2\mathbb{R}$ be a representation which takes the generators to a pair of hyperbolic isometries of $\mathbb{H}^2$ with orthogonal axes, each having translation length t. For t large enough, this is a discrete, faithful representation. What is the Morgan-Shalen limit of this path (as $t \to \infty$) in the boundary of the $\mathbb{F}_2$ character variety? What action on a tree arises from the Bestvina-Paulin geometric limit construction, using the intersection of axes of the generators as a base point?

10. (*) When studying projective structures on punctured Riemann surfaces, a natural condition that makes the theory analogous to the compact case is to consider only *bounded projective structures*, which differ from a finite-area hyperbolic structure by a holomorphic quadratic differential with at most simple poles at the punctures.

    (a) Consider a four-punctured sphere $X_\lambda = \mathbb{C} \setminus \{0, 1, \lambda\}$. This Riemann surface inherits a projective structure as a subset of $\hat{\mathbb{C}}$, which we call the *embedded projective structure*. Is this projective structure bounded?

    (b) Consider the difference between an arbitrary bounded projective structure on X_λ and the embedded projective structure. What family of meromorphic quadratic differentials on X_λ arises in this way?

        (Hint: It is not just the space of differentials with at most simple poles at the punctures!)

    (c) Generalize (b) to spheres with n punctures and the differences between their embedded and bounded projective structures.

2 Open Problem Session

Gunning's survey [8] on projective structures identifies two key problems in the theory:

A. Determine which representations arise as holonomy of projective structures. (That is, determine the *image* of the holonomy map.)

B. Given a representation of a surface group in $\mathrm{PSL}_2\mathbb{C}$, determine which projective structures have this as their holonomy representation.
(That is, determine the *fibers* of the holonomy map.)

While there has been significant progress on these questions, variations and extensions of them continue to guide a lot of ongoing work. Here are some related open problems:

1. In [6], Gallo, Kapovich, and Marden answer question (A) for the space of all projective structures on surfaces of a given topological type: A representation arises as a projective holonomy if and only if it is nonelementary and can be lifted to $\mathrm{SL}_2\mathbb{C}$.

   However, comparatively little is known about the corresponding fiberwise question: Given a compact Riemann surface X, and a representation $\rho : \pi_1 X \to \mathrm{PSL}_2\mathbb{C}$ (say, described by specifying images for a finite generating set), decide whether or not there is a projective structure on X with holonomy ρ.

2. In [7], Goldman showed that for a quasi-Fuchsian representation, there is a natural description of the fiber of the holonomy map in terms of the "wrapping" of the developing map relative to the limit set. The fiber is in bijection with the set of isotopy classes of multicurves (a countably infinite set).

   Can this picture be extended to some points on the boundary of quasi-Fuchsian space? For example, can one analogously classify the projective structures with holonomy that is a cyclic cover of a fibered 3-manifold? (I learned of this question from François Labourie.)

3. The holonomy map from the space of projective structures to the character variety is a local diffeomorphism but it is not a covering map (as shown by Hejhal [9]). M. Kapovich has posed (in [10]) the problem of quantifying the failure to be a covering by giving a criterion to identify when a path of holonomy representations can be lifted to a path of projective structures.

Here are some additional open problems and topics of ongoing research related to the mini-course material:

4. The characterization of holonomy limits of projective structures on a fixed Riemann surface involves the possibility of "folding" of the dual tree of the quadratic differential. Does nontrivial folding actually occur? That is, does there exist a divergent sequence of projective structures whose Morgan-Shalen limit point is not equal to the dual tree of the limit Schwarzian?

   (It seems likely that this can happen. It would be natural to first study the local problem—families of polynomial quadratic differentials with colliding zeros, and the corresponding developing maps.)

5. After fixing a Riemann surface structure X, one can identify the $\mathrm{SL}_2\mathbb{C}$ character variety of the underlying topological surface with the moduli space of rank-2 Higgs bundles with trivial determinant. The projective structures on X determine a special locus within the character variety—what is the corresponding set within the space of Higgs bundles?

   (This question is essentially problem (1) above, but translated from characters to Higgs bundles. This set of Higgs bundles has been studied from other perspectives and is known as the *brane of Opers*, e.g. in [14, Sec. 4.6]. However, an explicit description of the Higgs bundles and Higgs fields comprising this set, such as is available for Teichmüller space, remains elusive.)

6. Because the action of $\mathrm{PSL}_2\mathbb{C}$ on $\mathbb{CP}^1$ preserves the set of round circles, there is a well-defined notion of a "circular curve" on a surface with a projective structure. Such curves can be seen as analogues of geodesics in this non-metric geometry.

   One can learn a lot about hyperbolic surfaces by cutting them along geodesics into simpler surfaces, or by studying the geometry of embedded metric disks (i.e. the injectivity radius function). Relatively little is known about the corresponding picture for circular curves and circular disks in surfaces with projective structures, and this seems like a fruitful direction for further exploration. Here are two concrete questions of this type:

   (a) Given an isotopy class γ of an essential simple closed curve on a surface S, describe the subset of the moduli space of $\mathbb{CP}^1$ structures on S in which γ can be realized by a circular curve.

   This question has many natural variations: We could ask the same for a collection of disjoint curves, or for a pants decomposition. The condition of realization by a circular curve could be weakened to a piecewise circular curve with k arcs, or a C^1 piecewise circular curve, etc.

(b) Kojima, Mizushima, and Tan have studied projective structures on compact surfaces that are *disk packings* (see e.g. [12]). This means that the surface can be decomposed into a finite collection of round disks with disjoint interiors such that the each connected component of the complement of the disks is triangular. Their survey article describes many conjectures and open questions related to such structures. Among them is the following "uniformization conjecture":

Show that for each isotopy class τ of triangulation of a surface S and for each Riemann surface structure $X \in \mathcal{T}(S)$, there is a unique projective structure on X in which τ is realized as the dual graph of a circle packing.

7. A key step in the work of Gallo-Kapovich-Marden on holonomy of projective structures (see [6]) is to show that for any nonelementary representation of $\pi_1 S$ into $\mathrm{PSL}_2\mathbb{C}$, there exists a *Schottky pants decomposition*, i.e. a pants decomposition of S in which the restriction of the representation to each pair of pants is a classical Schottky group. It is then natural to ask for a characterization of the set of Schottky pants decompositions that a given representation admits, and how this set varies as the representation is moved through the character variety.

More generally, one could start with a representation $\pi_1 S \to G$ into an a Lie group G and ask what conditions on a representation of $\mathbb{F}_2 \to G$ can be ensured for the restriction of ρ to each element of some pants decomposition. (Perhaps this could be the starting point for a program to understand a generic representation of $\pi_1 S$ into G in the same way that the Gallo-Kapovich-Marden theorem says that one can use $\mathbb{CP}^1$ structures to understand a generic representation of $\pi_1 S$ into $\mathrm{SL}_2\mathbb{C}$.)

3 Bibliography

Some of the references below are cited in the exercises and open problems discussed above.

In addition, [4] and [8] are surveys that (along with their references) could serve as a starting point for learning about projective structures.

The mini-course lectures focused on outlining the main results of [3] (and introduction enough of the theory of $\mathbb{CP}^1$ structures to put those results in context). Another exposition of this material can be found in [15].

Thurston's description of the Schwarzian derivative in terms of osculating Möbius transformations can be found in [18]. This picture is substantially expanded and formalized in terms of projective connections in the thesis of Anderson [1], which unfortunately has not been published in any other form.

Epstein surfaces were introduced in [5]. In addition to their use in the holonomy limits construction described in the mini-course, these surfaces have been applied to problems in Kleinian groups related to cone-manifold deformations [2], Skinning maps [11] [16], and renormalized volumes of geometrically finite hyperbolic manifolds [17], [13].

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