Non-uniqueness of minimal surfaces

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Minimal maps in products of Riemann Surfaces

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given as the sum of energies of the corresponding harmonic diffeomorphisms.

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Uniqueness Conjecture

For each Fuchsian representation $\rho : \pi_1(\Sigma_{\mathbf{g}}) \to \prod_{i=1}^{n} \mathbf{PSL}(2, \mathbb{R})$, the energy functional E_ρ has a unique stationary point (and thus a unique minimal surface).

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- Schoen proved (applying the work of Micallef-Wolfson) that this is the only stationary point of E_ρ providing $n = 2$.

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Then the inequality

$$\operatorname{Re} \sum_{i=1}^{n} \int_S \phi_i \frac{\mu_i}{1 - |\mu_i|^2} \leq \sum_{i=1}^{3} \int_S |\phi_i| \frac{|\mu_i|^2}{1 - |\mu_i|^2}$$

holds where

$$\mu_i = \frac{(f_i)_{\bar{z}}}{(f_i)_z} \frac{d\bar{z}}{dz}.$$

Equivalence between the conjectures

Equivalence Theorem

The "Uniqueness Conjecture" is equivalent to the "Generalized Main Inequality Conjecture" for each n.

Equivalence between the conjectures

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$$\int_S \phi \frac{\mu_1}{1-|\mu_1|^2} - \int_S \phi \frac{\mu_2}{1-|\mu_2|^2} \leq \int_S |\phi| \frac{|\mu_1|^2}{1-|\mu_1|^2} + \int_S |\phi| \frac{|\mu_2|^2}{1-|\mu_2|^2}$$

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- This new inequality provides a new proof of the Teichmüller theorem about the uniqueness of geodesics between two points in $\mathbf{T_g}$.

The New Main Inequality fails when $n = 3$

Counterexample Lemma

For every large enough $\mathbf{g} \geq 2$*, there exist*

- *closed Riemann surfaces* S *and* S' *of genus* $\mathbf{g}$*,*
- *mutually homotopic diffeomorphisms* $f_i : S \to S'$*,* $i = 1, 2, 3$*,*
- *holomorphic quadratic differentials* ϕ_i *on* S*, satisfying the condition*

$$\phi_1 + \phi_2 + \phi_3 = 0,$$

such that the following strict inequality

$$\operatorname{Re} \sum_{i=1}^{3} \int_S \phi_i \frac{\mu_i}{1 - |\mu_i|^2} > \sum_{i=1}^{3} \int_S |\phi_i| \frac{|\mu_i|^2}{1 - |\mu_i|^2}.$$

Non-uniqueness of minimal surfaces

Non-uniqueness Theorem

For every large enough $\mathbf{g} \geq 2$*, there exists a Fuchsian representation* $\rho : \pi_1(\Sigma_{\mathbf{g}}) \to \prod_{i=1}^{3} \mathbf{PSL}(2, \mathbb{R})$ *such that* $\mathsf{E}_\rho : \mathbf{T}_{\mathbf{g}} \to (0, \infty)$ *has at least two stationary points.*

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- We also give a different proof of the uniqueness result when $n = 2$ (one can also give a new proof that the energy functional is pluri-subharmonic on $\mathbf{T}_{\mathbf{g}}$).

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- Fix Riemann surfaces S, S', diffeomorphisms $f_i : S \to S'$, and holomorphic quadratic differentials ϕ_i satisfying the assumptions and conclusions of the lemma.
- For every $t > 0$, we let (M_i^t, σ_i^t) be the hyperbolic Riemann surface, and $h_i^t : S \to (M_i^t, \sigma_i^t)$ the harmonic diffeomorphism, such that $\text{Hopf}(h_i^t) = t\phi_i$.

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- Fix Riemann surfaces S, S', diffeomorphisms $f_i : S \to S'$, and holomorphic quadratic differentials ϕ_i satisfying the assumptions and conclusions of the lemma.
- For every $t > 0$, we let (M_i^t, σ_i^t) be the hyperbolic Riemann surface, and $h_i^t : S \to (M_i^t, \sigma_i^t)$ the harmonic diffeomorphism, such that $\mathrm{Hopf}(h_i^t) = t\phi_i$.
- For each fixed t, we consider the energy functional $\mathrm{E}^t : \mathbf{T_g} \to \mathbb{R}$.

Non-uniqueness of minimal surfaces

- Note that S is the critical point of each function E^t because

$$\mathrm{Hopf}(h^t) = t(\phi_1 + \phi_2 + \phi_3) = 0,$$

where $h^t : S \to M_1^t \times M_2^t \times M_3^t$ is the corresponding product map (that is, h^t is a minimal surface).

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- The idea of this proof is to show that for t large enough E^t does not achieve its minimum at S. Thus, the global minimum of E^t provides another critical point of E^t which proves the Non-uniqueness Theorem.

Non-uniqueness of minimal surfaces

- The inequality

$$\mathrm{E}^t(S') \leq \sum_{i=1}^{3} \mathcal{E}(h_i^t \circ f_i^{-1})$$

holds because the total energy of the diffeomorphism $h_i^t \circ f_i^{-1} : S' \to M_i^t$ is greater or equal than the energy of the harmonic diffeomorphism in its homotopy class.

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- This shows

$$\mathrm{E}^t(S') - \mathrm{E}^t(S) \leq \sum_{i=1}^{3} \left(\mathcal{E}(h_i^t \circ f_i^{-1}) - \mathcal{E}(h_i^t)\right)$$

$$\leq -4\,\mathrm{Re} \sum_{i=1}^{3} \int_S t\phi_i \frac{\mu_i}{1-|\mu_i|^2} + 2 \sum_{i=1}^{3} \int_S \mathbf{e}(h_i^t) \frac{|\mu_i|^2}{1-|\mu_i|^2}.$$

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- Dividing the previous inequality by $4t$ yields

$$\frac{\mathrm{E}^t(S') - \mathrm{E}^t(S)}{4t} = \sum_{i=1}^{3} \left(-\operatorname{Re} \int\limits_S \phi_i \frac{\mu_i}{1 - |\mu_i|^2} + \int\limits_S |\phi_i| \frac{|\mu_i|^2}{1 - |\mu_i|^2} \right) + o(1),$$

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- From the lemma we see that when $o(1)$ is sufficiently small, the left hand side is negative. Thus, $\mathrm{E}^t : \mathbf{T_g} \to (0, \infty)$ does not achieve its global minimum at S which proves the theorem.

Generalized Main Inequality on $\mathbb{D}$

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- These functions ϕ_i are squares of certain quadratic polynomials, and μ_i's are then constructed so that inequality holds.

The Infinitesimal Version

Infinitesimal Main Inequality Conjecture

Given mutually infinitesimally equivalent $\dot{\mu}_i \in L^\infty(\mathbb{D}) \cap C_0^\infty(\mathbb{D})$, *and functions* $\phi_i \in \mathcal{H}^1(\mathbb{D})$, $i = 1, ..., n$, *with the property*

$$\phi_1 + \cdots + \phi_n = 0,$$

then the inequality

$$-\operatorname{Re} \sum_{i=1}^{n} \int_{\mathbb{D}} \phi_i \dot{\mu}_i T(\dot{\mu}_i)\, dxdy \leq \sum_{i=1}^{n} \int_{\mathbb{D}} |\phi_i||\dot{\mu}_i|^2\, dxdy, \qquad (\star)$$

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- This conjecture is equivalent to the Uniqueness Conjecture for minimal surfaces in a product of n Riemann surfaces.

The case $n = 2$

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Proposition

Suppose $\dot\mu, \dot\nu \in L^\infty(\mathbb{D})$ are infinitesimally equivalent, and let $\psi \in \mathcal{H}^2(\mathbb{D})$. Then

$$\int_{\mathbb{D}} \psi^2 \dot\mu T(\dot\mu)\, dxdy - \int_{\mathbb{D}} \psi^2 \dot\nu T(\dot\nu)\, dxdy$$

$$= \int_{\mathbb{D}} (\psi\dot\mu) T(\psi\dot\mu)\, dxdy - \int_{\mathbb{D}} (\psi\dot\nu) T(\psi\dot\nu)\, dxdy.$$

The case $n = 2$

Proof.

$$\int_{\mathbb{D}} \psi^2 \dot{\mu} T(\dot{\mu})\, dxdy - \int_{\mathbb{D}} (\psi\dot{\mu}) T(\psi\dot{\mu})\, dxdy =$$

$$\frac{1}{2} \int_{\mathbb{D}\times\mathbb{D}} \frac{(\psi(z) - \psi(\zeta))^2}{(z-\zeta)^2} \dot{\mu}(z)\dot{\mu}(\zeta) = \frac{1}{2} \int_{\mathbb{D}\times\mathbb{D}} \frac{(\psi(z) - \psi(\zeta))^2}{(z-\zeta)^2} \dot{\nu}(z)\dot{\nu}(\zeta)$$

$$= \int_{\mathbb{D}} \psi^2 \dot{\nu} T(\dot{\nu})\, dxdy - \int_{\mathbb{D}} (\psi\dot{\nu}) T(\psi\dot{\nu})\, dxdy.$$

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$$\sum_{i=1}^{2} \int_{\mathbb{D}} \psi_i^2 \dot{\mu}_i T(\dot{\mu}_i)\, dxdy = \sum_{i=1}^{2} \int_{\mathbb{D}} (\psi_i \dot{\mu}_i) T(\psi_i \dot{\mu}_i)\, dxdy.$$

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- Then using the Cauchy-Schwartz inequality, and then the fact that the Beurling Transform T is an L^2-isometry, we get

$$\left| \sum_{i=1}^{2} \int_{\mathbb{D}} (\psi_i \dot{\mu}_i) T(\psi_i \dot{\mu}_i)\, dxdy \right| \leq \sum_{i=1}^{2} \int_{\mathbb{D}} |\psi_i \dot{\mu}_i| |T(\psi_i \dot{\mu}_i)|\, dxdy$$

$$\leq \sum_{i=1}^{2} \int_{\mathbb{D}} |\psi_i|^2 |\dot{\mu}_i|^2\, dxdy$$

The general case

- In general (for every n), we obtain using the Stokes theorem

$$\sum_{i=1}^{n} \int_{\mathbb{D}} \psi_i^2 \dot{\mu}_i T(\dot{\mu}_i)\, dxdy = \sum_{i=1}^{n} \int_{\mathbb{D}} \big(\psi_i P(\dot{\mu}_i)\big)_{\bar{z}} \big(\psi_i P(\dot{\mu}_i)\big)_{z}\, dxdy.$$

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- In general (for every n), we obtain using the Stokes theorem

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- Thus, the $(\star)$ inequality becomes

$$0 \leq \sum_{i=1}^{n} \int\limits_{\mathbb{D}} |(f_i)_{\bar{z}}|^2\, dxdy + \operatorname{Re} \sum_{i=1}^{n} \int\limits_{\mathbb{D}} (f_i)_{\bar{z}} \big(f_i\big)_{z}\, dxdy, \qquad (\star\star)$$

where $f_i = \psi_i P(\dot{\mu}_i)$.

The general case

- For a smooth function $f : \mathbb{D} \to \mathbb{C}$, we define the functional

$$\mathcal{F}(f) = \int_{\mathbb{D}} |f_{\bar{z}}|^2 \, dxdy \; + \; \mathrm{Re} \int_{\mathbb{D}} f_z f_{\bar{z}} \, dxdy.$$

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- Then the $(\star\star)$ can be written as

$$0 \leq \sum_{i=1}^{n} \mathcal{F}\big(\psi_i P(\dot{\mu}_i)\big) \qquad\qquad (\star\star)$$

The general case

Lemma 1

Suppose $\psi(z) = a + bz + cz^2$ *is a quadratic polynomial which has no zeroes on the unit circle* $\partial\mathbb{D}$. *Then for every* $\epsilon > 0$, *there exists* $\dot{\mu} \in L^{\infty}(\mathbb{D}) \cap C_0^{\infty}(\mathbb{D})$ *such that*

$$P(\dot{\mu})(z) = \frac{1}{z}, \quad z \in \mathbb{D}^*,$$

and

$$\left|\mathcal{F}\big(\psi P(\dot{\mu})\big) - \pi\big(|a|^2 + \mathrm{Re}(ac)\big)\right| \leq \epsilon.$$

Three polynomials

- Define the following polynomials of order two

$$\psi_1 = \mathbf{i} - 5z + \mathbf{i}\frac{25}{4}z^2, \quad \psi_2 = \mathbf{i} + 5z + \mathbf{i}\frac{25}{4}z^2, \quad \psi_3 = -\sqrt{2} + \sqrt{2}\frac{25}{4}z^2.$$

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$$\psi_1^2 + \psi_2^2 + \psi_3^2 \equiv 0,$$

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- Then

$$\psi_1^2 + \psi_2^2 + \psi_3^2 \equiv 0,$$

and

$$\pi \sum_{i=1}^{3} \left(|a_i|^2 + \mathrm{Re}(a_i c_i) \right) = -21\pi,$$

where $\psi_i(z) = a_i + b_i z + c_i z^2$, $i = 1, 2, 3$.

Three polynomials

- We now choose $\dot{\mu}_i$'s from Lemma 1 (we let $\epsilon = \pi$ in the lemma), and get

$$\sum_{i=1}^{n} \int_{\mathbb{D}} |(f_i)_{\bar{z}}|^2 \, dxdy + \mathrm{Re} \sum_{i=1}^{n} \int_{\mathbb{D}} (f_i)_{\bar{z}} \overline{(f_i)}_z \, dxdy < -20\pi,$$

where as before $f_i = \psi_i P(\dot{\mu}_i)$.

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- Since $f_{\bar{z}} \equiv a$, and $f_z \equiv c$, we easily compute

$$\mathcal{F}(f) = \pi \left(|a|^2 + \mathrm{Re}(ac) \right).$$

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