François LABOURIE

Université Côte d'Azur

Hitchin representation

S = closed connect oriented surface genus $g \geqslant 2$.

$$\begin{tikzcd}
\rho : \pi_1(S) \arrow[rr, "n\text{-Fuchsian}"] \arrow[dr, "\text{fuchsian}"'] & & PSL_n(\mathbb{R}) \\
& PSL_2(\mathbb{R}) \arrow[ur, "\text{irreducible representation}"'] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

A Hitchin representation is a representation that can be deformed in an n-Fuchsian

Hitchin component

$\mathcal{H}(n) = \{$ Hitchin representation $\}$ connected component of $\mathrm{Rep}(\pi_1(S), PSL_n(\mathbb{R}))$

$\mathcal{H}(2) =$ Teichmüller space

Hitchin representations

Nice dynamical properties

$\rightsquigarrow$ associated to a "geodesic flow"

$\rightsquigarrow$ length spectrum, entropy etc...

$\rightsquigarrow$ Mapping class group acts properly on $\mathcal{H}(n)$

$\rightsquigarrow$ $\forall \gamma \in \pi_1(S) \setminus \{1\}$, $\rho(\gamma)$ is $\mathbb{R}$-split

Hitchin component

behaves a lot like Teichmüller component $SL(2)$ from the dynamical point of view.

$$\boxed{\text{What about uniformisation ?}}$$

Hitchin parametrisation

$X = (S, J)$ be a Riemann surface

$Q^n_J(S) := \{ n\text{-holomorphic diff on } X \}$

$$\boxed{\bigoplus_{n=2}^{m} Q^n_J(S) \overset{\Psi}{\longleftrightarrow} \mathcal{H}(m)}$$

break the Mapping class group symmetry

det $\mathcal{E}$ = bundle of Teichmüller

$$\mathcal{E}_J = \bigoplus_{n=3}^{m} Q_J^n(S)$$

We have a MCG equivariant

$$\Phi : \mathcal{E} \longrightarrow \mathcal{H}(n)$$

same dimension

- $\mathcal{E}$ = {space of equivariant minimal surface in M_m under Hitchin representations}.
- $\mathcal{E}$ has a Kähler structure

  [I. Kim - G. Zhang, L]
- Φ is surjective

Conjecture

is Φ injective ?

is th a omplex interpretation of $\mathcal{H}(n)$?

is $\mathcal{H}(n)/M$ a vector bundle over Riemann moduli space ?

(J. W with Sambarino)

Hitchin representation

$\hookrightarrow$ geodesic current = a locally finite $\pi_1(S)$-invariant measure on $\partial_\infty \pi_1(S)^2$

. —— .

If X = Riemann surface

$$\langle \nu, X \rangle_m(z) := \int_{(\partial_\infty \mathbb{H}^2)^2} \left[\frac{1}{(z-x)} - \frac{1}{(z-y)} \right] d\nu(x,y)\, dz^m$$

Given ν there is a unique J so that $\langle \nu, X \rangle_2 (z) = 0$

$$\mathcal{H}(n) \longrightarrow \text{Teich}$$

Is this a fibration? are the fibers contractible? what are the meanings of $\langle \nu | J \rangle_m$?

Thm :

is there a complex interpretation of $\mathcal{H}(n)$?

—

is $\mathcal{H}(n)/_{MCG}$ a vector bundle over Riemann moduli space?

Fock-Thomas higher complex structures.

A question (not unrelated)

Let M be a manifold, i a function on $M\times M$ so that

(x,x) is a minimum of $y \to i(x,y)$

assume $i(x,y) = i(x,y)$

Say i is W if it enjoys the following property:

the pairing $T_xM \times T_yM \longrightarrow \mathbb{R}$

$(u,v) \longmapsto \partial_y[(\partial_x i(x,y))u]v$ is non

degenerate. To be more precise. let :

let $f^y : x \mapsto i(x,y)$

let $g^u : y \mapsto D_x f^y(u)$

And $\langle u|v\rangle = D_y g^u(v)$. $\nearrow$ $i =$ intersection on $\mathcal{T}$

The question is does i has the W property?

that would imply a new parametrisation of $\mathcal{T}$ by quadratic diff.