The mathematical focus of the GEAR Network

1. Introduction

The mathematical focal point of the GEAR network is the interplay of topology, geometry, and dynamics on character varieties. The core mathematics, namely the theory of locally homogeneous geometric structures, flat bundles, and their deformation spaces has a long and distinguished history. The subject has recently acquired a markedly multifaceted flavor with insights and techniques from several distinct sources being applied to emerging problems.

The moduli space of flat bundles (sometimes called a representation variety, character variety, or a deformation space) is a central object in several mathematical fields. Closely related is the deformation space of locally homogeneous geometric structures. These spaces parametrize equivalence classes of representations of the fundamental group of a space Σ on one hand, and flat connections on bundles over Σ on the other. They often arise as spaces of solutions to basic geometric problems, and their global properties provide powerful topological invariants. The ubiquity of these spaces demands the interplay of methods and viewpoints from multiple mathematical communities. These communities include: *Classical Teichmüller theory, Kleinian groups,* 3*-manifold topology, Moduli of vector bundles and gauge theory, Mathematical physics, Dynamical systems, Invariant theory, and Geometric PDE's.*

2. Historical roots

Classically, moduli spaces arose in the study of monodromy of systems of analytic differential equations. Analytic continuation of solutions of a system led to a linear representation of the fundamental group. These were the first examples of local systems, or vector bundles with flat connections. It was soon realized that these objects were closely related to the uniformization of Riemann surfaces, providing an analytic motivation for the study of discrete groups (Fuchsian groups, Kleinian groups).

Uniformization identifies Riemann surfaces with spherical, Euclidean or hyperbolic surfaces and this leads to two avenues for studying the Teichmüller space, $\mathfrak{T}(\Sigma)$, of (marked) conformal structures (or constant curvature metrics) on a surface Σ. Two largely independent schools of research developed around these approachs. The first primarily utilized techniques from complex analysis while the other focused on hyperbolic geometry and discrete groups of isometries.

The modern theory of Teichmüller theory began with the work of Ahlfors and Bers. In their hands quasiconformal mappings became a powerful tool for investigating the holomorphic geometry of $\mathfrak{T}(\Sigma)$. An important example is the *Simultaneous Uniformization* theorem of Bers [12]. Every Riemann surface of genus $g > 1$ has a unique *hyperbolic structure,* that is, can be represented as a quotient $H^2/\rho(\pi_1(\Sigma))$, where $\pi_1(\Sigma) \xrightarrow{\rho} \mathsf{PSL}(2,\mathbb{R})$ is a discrete embedding (a *Fuchsian representation*). This action extends to $\mathbb{C}\mathsf{P}^1 \approx S^2$, preserving a circle. A Fuchsian representation can be deformed within $\mathsf{PSL}(2,\mathbb{C})$ to an action on $\mathbb{C}\mathsf{P}^1$ that preserves a fractal nonrectifiable Jordan curve Λ. Such an action is quasiconformally conjugate to a Fuchsian action, taking Λ to the invariant circle. Bers showed that these *quasi-Fuchsian* groups have a beautiful and easily stated parametrization: they are parametrized by the two marked Riemann surfaces that are the quotient of $\mathbb{C}\mathsf{P}^1 - \Lambda$ by the group action. In particular, the deformation space identifies naturally with the product $\mathfrak{T}(\Sigma) \times \overline{\mathfrak{T}(\Sigma)}$.

These representations correspond to hyperbolic 3-manifolds homeomorphic to $\Sigma \times \mathbb{R}$, which can be naturally compactified to $\Sigma \times [-\infty, \infty]$ using the compactification of H^3 as a closed ball. These ideas were later extended to other 3-manifolds by Maskit, Marden and others to describe the deformation space of hyperbolic structures of a certain nice type ("geometrically finite manifolds") explicitly in terms of conformal structures on ideal boundary components.

At roughly the same time, Mostow [92] proved his *strong rigidity theorem,* that for closed hyperbolic manifolds of dimension ≥ 3, every homotopy-equivalence is homotopic to a unique *isometry.* This was extended to finite volume complete manifolds by Prasad several years later.

The stage was set for Thurston's revolutionary work in the late 1970's on hyperbolic 3-manifolds. The topological theory of 3-manifolds had provided basic decomposition theorems due to Kneser, Milnor (for connected sums) and Jaco, Shalen and Johannson (for sums along tori). Thurston realized that the basic building blocks in these decomposition theorems should have locally homogeneous Riemannian structures. He proceeded to prove this *Geometrization Conjecture* for a large class of 3-manifolds. The deformation theory of Kleinian groups, based on simultaneous uniformization, played an important role in Thurston's proof.

Thurston's work in dimension 3 crucially involved ideas from his work in dimension 2. He introduced fundamental objects, measured foliations and measured geodesic laminations, that provide both infinitesimal and asymptotic versions of deformations of hyperbolic structures on surfaces [14]. Although his work was based in hyperbolic geometry and topology, it was deeply connected to objects like quadratic differentials that are central to the complex analytic study of Teichmuller space. This helped reunite the two strands of Teichmuller theory.

The *Mapping Class Group* $\mathsf{Mod}(\Sigma)$, the group of diffeomorphisms of a surface up to isotopy, acts on Teichmuller space $(\mathfrak{T}(\Sigma))$ with quotient equal to the *moduli space* of structures. Thurston defined a $\mathsf{Mod}(\Sigma)$-invariant compactification of $\mathfrak{T}(\Sigma)$, which led to a classification of elements of $\mathsf{Mod}(\Sigma)$, analogous to Jordan normal form for matrices [106].[44]. This demonstrated both the power of using dynamics to study spaces of geometric structures and the degree to which such spaces provide interesting examples of groups actions. Furthermore, the space of (projectivized) measured foliations which forms the boundary of this compactification has a natural simplicial structure. Combinatorial structures are a powerful tool for studying both the space $\mathfrak{T}(\Sigma)$ and the group $\mathsf{Mod}(\Sigma)$. Finding combinatorial objects to associate to other deformation varieties, is an important goal of current research.

Another significant development came from completely different direction. Inspired by gauge theory for the weak interaction, Atiyah, Singer and Hitchin [5] began the study of $SU(2)$-instantons, and quickly realized that this theory provided an analytic framework for the theory of stable vector bundles in algebraic geometry. Soon the analytic cornerstones were supplied by the work of Uhlenbeck [112] and Taubes [105], which led to Donaldson's startling discovery [40] of the vast difference between the smooth and topological categories in dimension 4 (coming on the heels of Freedman's spectacular work on topological 4-manifolds).

In dimension 2, Atiyah and Bott [4] used gauge theory to give a new understanding of the moduli space of stable vector bundles over Riemann surfaces, which by the work of Narasimhan-Seshadri [93] corresponds to surface group representations in compact Lie groups. Using Morse theory on the space of connections, Atiyah and Bott computed the cohomology of the moduli space. Shortly thereafter, the theory of *Higgs bundles,* developed by Hitchin [67] and Simpson [102, 103] extended these techniques to all real reductive Lie groups. This theory combined the moduli theory of vector bundles (the nonabelian extension of classical Abel-Jacobi theory [60]) with the theory of harmonic maps of Riemannian manifolds. Over a Riemann surface, representations of $\pi_1(\Sigma)$ acquire an analytic interpretation, for which the topology of their moduli can be computed using Hodge theory and algebro-geometric machinery.

3. Current Problems and Future Interactions

Distinct research communities that evolved from the historical roots are re-united by the GEAR network. We describe recent significant interactions, as well as promising areas for future collaborations between these communities.

3.1. **Hyperbolic 3-manifolds.** Thurston's analysis of hyperbolic 3-manifolds focused attention on several key problems, old and new, that have only recently been solved, including the ending lamination conjecture (ELC), the density conjecture, and the tameness conjecture. These recent successes suggest that it might be possible to employ similar techniques or to search for similar key constructions in the context of other representation spaces or spaces of geometric structures. A central motivation for this network is to bring together experts in hyperbolic geometry and 3-manifolds with those who study other types of geometric structures.

The tameness conjecture says that any hyperbolic 3-manifold with finitely generated fundamental group is homeomorphic to the interior of a compact 3-manifold. Proved by Agol [1] and Calegari-Gabai [28], it is a question of controlling the topology of the ends of the manifold. This is done by finding certain types of surfaces that fill out the ends. Such a tameness question is natural for other non-compact, geometric manifolds (see section 3.7).

The density conjecture says that the convex cocompact hyperbolic structures on a hyperbolic 3-manifold are dense in the space of all hyperbolic structures. The convex cocompact structures have compact convex cores and are parametrized by conformal structures at infinity. Another natural question was how to parametrize the remaining structures, those with non-compact convex cores. The ELC states that these are parametrized by geodesic laminations, one for each end. This was proved by Brock, Canary and Minsky [22, 88] using geometric model manifolds that are combinatorially defined, depend only on a geodesic lamination determined by each end, and are then shown to be quasi-isometric to the actual hyperbolic manifold. The main point is that the models are combinatorially determined by something similar to a point at infinity in Thurston's compactification of $\mathfrak{T}(\Sigma)$.

Another important line of research comes from Thurston's theory of hyperbolic Dehn surgery, whereby infinitely many closed manifolds approximate a noncompact finite volume, complete hyperbolic manifold ([110]). Although Mostow-Prasad rigidity says that finite volume, complete structures are unique, in dimension 3 there is a nontrivial deformation space of incomplete structures in the noncompact case. The nearby closed manifolds are obtained by completing special incomplete ones but most of the completions are singular. The deformation theory of singular hyperbolic structures has become an important tool in studying representation varieties ([70],[71],[114]). It played an important role Brock-Bromberg's proof of the density conjecture ([24], [23]) and in the proof of the Orbifold Theorem ([13], [32]). Singular structures in other geometries (see section 3.8) lie at a natural interface between groups in the network.

3.2. **Character varieties and 3-manifolds.** Pioneering work of Culler and Shalen [34] made $\mathsf{SL}(2,\mathbb{C})$-character varieties an important tool in 3-dimensional topology. The algebraic geometry of the character variety led to new invariants of 3-manifolds. The basic idea was that the ideal points of curves in these affine varieties led to actions of the fundamental group on simplicial trees (and $\mathbb{R}$-trees for higher-dimensional subvarieties). This led to splitting theorems and hence incompressible surfaces, which is the necessary ingredient for the Haken-Waldhausen machinery. By using this theory for $\mathbb{R}$-trees, Morgan and Shalen [90, 91] gave new proofs of Thurston's acylindricity theorem [108], and an algebraic interpretation of the Thurston cone of measured geodesic laminations on a surface. One of the most spectacular applications of this theory is the *Cyclic surgery theorem,* of Culler-Gordon-Luecke-Shalen [33], giving strong restrictions on which surgeries on a knot can give 3-manifolds with cyclic fundamental group.

3.3. **Special Representations.** Surface group representations into groups of Hermitian type enjoy an integer-valued characteristic class invariant, whose maximization implies strong discreteness and rigidity properties ([51, 53], [111], [17, 18], [26, 27]). Using Higgs bundles, Hitchin [69] found connected components of representations into split real forms which were topological cells, containing $\mathfrak{T}(\Sigma)$. Labourie [76] and Guichard [62] found dynamical characterization of these Hitchin components, while Fock and Goncharov [45], seeking to quantize these spaces, related these components to Lusztig positivity [80] and the emerging theory of cluster algebras [47].

A key idea in this subject, called *"higher Teichmüller theory,"* by Fock-Goncharov, is that a generalization of Penner's shearing coordinates [94] gives coordinates on the Hitchin components. The combinatorial nature of the space of these coordinates makes them particularly useful. Also, the natural (Weil-Petersson) symplectic structure has a particularly tractable form in these coordinates, which enables the quantization of these spaces. A natural question is to define other global coordinate systems on components of special representations of surface groups, say, analogous to the Fenchel-Nielsen coordinates on $\mathfrak{T}(\Sigma)$.

For $G = \mathsf{SL}(3,\mathbb{R})$ these components correspond to the deformation space of *convex* $\mathbb{R}\mathsf{P}^2$-*structures,* studied by Choi and Goldman [30], Loftin [78, 79], Labourie [75], Benoist [7] -[11] and others. Interpreting the Hitchin components for other split real forms as geometric structures is an important question, and has been achieved for $\mathsf{SL}(4,\mathbb{R})$ in recent work of Guichard-Wienhard [63]. Labourie [76] defined a general class of representations (*Anosov representations,* since they define Anosov extensions of the geodesic flow to the corresponding flat bundles) which include representations in the Hitchin component as well as maximal representations. These results and techniques lead to the properness of the $\mathsf{Mod}(\Sigma)$-action on these special representations. Recent work of Guichard and Wienhard [64] makes significant inroads in relating these special classes of representations to geometric structures on compact manifolds.

3.4. **Topology of character varieties.** In [37, 37] Daskalopoulos, Weitsman, Wentworth and Wilkin compute the cohomology of the $\mathsf{SL}(2,\mathbb{C})$-character variety of a closed surface, a significant achievement for the simplest complex semisimple Lie group. For the unitary groups $\mathsf{U}(p,q)$ and many other groups of Hermitian type, Bradlow, Garcia-Prada, Gothen, Mundet and others ([17, 18]) have produced sharp results on the topology. Using techniques of boundary maps and Anosov representations, Guichard and Wienhard [65] have given lower bounds on the number of connected components, constructed special model representations and determines the possible holonomies. The parallels between these two approaches deserve further study.

3.5. **Variation of the moduli space with the Riemann surface.** For a complex reductive group $G_{\mathbb{C}}$ and Riemann surface M the Higgs bundle moduli space $\mathfrak{H}(M, G_{\mathbb{C}}) := \mathsf{Hom}(\pi_1(M), G_{\mathbb{C}})//G_{\mathbb{C}}$ carries a natural hyper-Kähler structure. While this structure varies with the complex structure on M, the underlying *complex-symplectic structure* on $\mathfrak{H}(M, G_{\mathbb{C}})$ depends only on the underlying topological surface Σ. For real forms $G \subset G_{\mathbb{C}}$, the corresponding moduli spaces inherit Kähler structures, and the underlying *symplectic structure* depends only on Σ. In particular this latter structure is invariant under the mapping class group $\mathsf{Mod}(\Sigma)$.

To analyze the dependence on the conformal structure, build a space $\mathfrak{H}(\Sigma, G_{\mathbb{C}})$ of Higgs bundles moduli spaces over marked Riemann surfaces, as the marked Riemann surface varies over the Teichmüller space $\mathfrak{T}(\Sigma)$. This family of hyper-Kähler manifolds with fixed symplectic structure is the natural object on which to investigate the dependence on the conformal structure of M. The quotient $\mathfrak{H}(\Sigma, G_{\mathbb{C}})/\mathsf{Mod}(\Sigma)$ is a flat orbi-bundle over $\mathfrak{T}(\Sigma)/\mathsf{Mod}(\Sigma)$, the Riemann moduli space. Pulling back to the unit tangent bundle, the horizontal lifts of the Teichmüller and Weil-Petersson geodesic flows to $\mathfrak{H}(\Sigma, G_{\mathbb{C}})/\mathsf{Mod}(\Sigma)$ define dynamical systems which are dynamically equivalent to the action of $\mathsf{Mod}(\Sigma)$-action on the character variety. Finer dynamical invariants should be more tractable in the context of the action of the simpler group $\mathbb{R}$.

3.6. **Actions of mapping class groups.** The action of the mapping class group on character varieties or deformation spaces is a rich source of interesting dynamical systems. For compact groups G, the action of $\mathsf{Mod}(\Sigma)$ on each component of $\mathsf{Hom}(\pi, G)/G$ is ergodic (Goldman [52],Pickrell-Xia [95, 96]), as is the $\mathsf{Out}(\pi)$-action on $\mathsf{Hom}(\pi, G)/G$ when π is a free group (Goldman [54] and Gelander [50]). For noncompact G, there are regions upon which the action is *proper.* In some case $\mathsf{Mod}(\Sigma)$ acts properly on entire components. However, in many cases the action of $\mathsf{Mod}(\Sigma)$ or $\mathsf{Out}(\pi)$ is proper on an open set whose closure contains subsets upon which the action displays nontrivial dynamics. Recently Minsky [89] has identified an open set in $\mathsf{Hom}(\mathbb{F}_n, \mathsf{SL}(2, \mathbb{C}))//\mathsf{SL}(2, \mathbb{C})$ upon which $\mathsf{Out}(\mathbb{F}_n)$ acts properly. Generalizing his notion of *primitive-stability* to arbitrary complex Lie groups is a natural and fundamental problem.

Geometric quantization as applied to moduli spaces [68] provides new insight on the representations and structure of the mapping class group through the work of Andersen [2], and others. Andersen's recent proof [3] that mapping class groups are Kazhdan crucially uses these moduli spaces.

Actions of subgroups of mapping class groups often behave very differently to those of the entire mapping class group. For example, by work of Masur [83] and McCarthy–Papadopoulos [84], these actions on the Thurston compactification of Teichmuller space resemble Kleinian groups acting on the compactified hyperbolic space. This has important consequences, for example, for the geometry of surface bundles [43, 73, 66]. The study of actions of subgroups on other character varieties is largely unexplored and is an important direction of future research.

3.7. **Affine, projective and flat Lorentzian structures.** Many of the surface group representations arise from the study of flat connections on manifolds. Indeed, $\mathbb{R}\mathsf{P}^2$-structures are just flat normal projective connections, and the basic problem of intepreting special representations in terms of geometric structures can be formulated gauge-theoretically. In particular many of the surface group representations into $\mathsf{Sp}(4, \mathbb{R})$ discussed by Guichard-Wienhard [65], Gothen [61], GarciaPrada-Gothen-Mundet [20] and Bradlow-GarciaPrada-Gothen [19] correspond to $\mathbb{R}\mathsf{P}^3$-structures on 3-manifolds with a natural contact geometry.

These symplectic representations also correspond to geometric structures modelled on the *Einstein universe,* a homogeneous space of $\mathsf{Sp}(4,\mathbb{R})$, which appears simultaneously as the conformal compactification of flat Minkowski $(2+1)$-space, the boundary of anti-de Sitter $3+1$-space and the Grassmannian of Lagrangian 2-planes in a 4-dimensional real symplectic vector space. Many of the special representations of surface groups also give rise to flat conformal Lorentzian 3-manifolds. See [6] for a description of the associated synthetic geometry.

Flat Lorentzian 3-manifolds arise from a much different context. The Bieberbach classification of Euclidean manifolds extends to a classification of *complete affine manifolds,* that is, closed manifolds with a geodesically complete flat torsionfree affine connection. In 1960, L. Auslander (incorrectly) claimed that such a manifold would necessarily have virtually polycyclic fundamental group. In 1977, Milnor asked whether Auslander's (still unproven) claim would be valid even without the assumption of compactness. Trying to prove Milnor's conjecture, Margulis [82] found counterexamples: free groups acting properly by affine transformations on $\mathbb{R}^3$. Such manifolds have natural flat Lorentz metrics. In his 1990 doctoral thesis [41], Todd Drumm found a geometric construction for this unexpected phenomenon.

By Fried-Goldman [49] the compact complete affine 3-manifolds were easily classified, but, due to the examples of Margulis-Drumm, the classification is lacking in the noncompact case. Recent progress by Charette, Drumm, Goldman, Labourie and Margulis ([29, 58]) suggests that the classification of all complete affine 3-manifolds with finitely generated fundamental group is imminent. The examples with nonsolvable fundamental group are conjecturally open solid handlebodies; this is equivalent to tameness (section 3.1).

This opens up many research directions: can a closed surface group admit a proper affine deformation in higher dimension? How do these isometrically flat spacetimes fit inside the space of conformally flat Lorentzian 3-dimensional manifolds? The geodesic flows on these flat spacetimes also provide new examples of dynamical systems closely related to the geodesic flows on associated hyperbolic surfaces. The relationship between geodesically complete flat affine manifolds in dimension 3 and complete hyperbolic 2-manifolds seems particularly striking.

3.8. **AdS manifolds in dimension** 3. In a seminal paper written in 1990, Geoff Mess [85] analyzed a class of 3-dimensional Anti-deSitter manifolds that are diffeomorphic to a compact surface Σ crossed with $\mathbb{R}$. He found a remarkable analogy between them and their hyperbolic counterparts, quasi-Fuchsian manifolds. As in the hyperbolic case, they are parametrized by a pair of conformal structures on Σ; in this case the parametrization comes from the identification of a component of the group of AdS isometries with $\mathsf{PSL}(2,\mathbb{R})\times\mathsf{PSL}(2,\mathbb{R})$ rather than structures at infinity. Furthermore, like a quasi-Fuchsian manifold, there is a compact convex core whose boundary consists of a pair of hyperbolic surfaces , each "bent" along a geodesic lamination. In the AdS, "bending" actually corresponds to "shearing" and using this Mess gave an AdS proof of Thurston's earthquake theorem that any two hyperbolic surfaces are connected by a unique earthquake (which is a shearing map) along a geodesic lamination.

This provides a tantalizing analogy between hyperbolic and AdS geometry in both this special case and for more general manifolds. For both the hyperbolic and the AdS structures there are conjectures about the existence and uniqueness of structures that are bent along certain pairs of laminations. Existence is known in the hyperbolic case, but uniqueness is only known is special cases. Recently, Bonsante and Schlenker [15] have proved existence in the AdS context by mimicking arguments in the hyperbolic situation. The GEAR network is an ideal setting for a project that investigates the hyperbolic and AdS conjectures simultaneously.

All closed AdS 3-manifolds are known to be Seifert fibered. Thus, it appears that AdS geometry should be relevant to a rather small segment of 3-dimensional topology. However, if one considers structures with singularities or on non-compact 3-manifolds, there are many more examples. For example, the flat Lorentzian examples discussed in section 3.7, which correspond to infinitesimal surface deformations, also lead to deformations into the space of AdS (and of 3-dimensional hyperbolic) structures.

3.9. **Transitional geometry.** "Transitional geometry" refers to the geometric description of a family of structures whose holonomy representations pass between different Lie groups. Passing from $\mathsf{SL}(2,\mathbb{R})$ into $\mathsf{SL}(2,\mathbb{C})$ is classical and has been discussed at length. Passing from $\mathsf{SL}(n,\mathbb{R})$ to $\mathsf{SL}(n,\mathbb{C})$ plays a role in the study of Higgs bundles. One can ask whether various theories like simultaneous uniformation, where complex representations correspond to a pair of real representations, hold for Hitchin components of $\mathsf{SL}(n,\mathbb{R})$ representations.

Another interesting situation arises when hyperbolic structures degenerate to lower dimensional structures, as occurs in Thurston's hyperbolic Dehn surgery theory or in the proof of the Orbifold theorem. For example a family of (possibly incomplete or singular) hyperbolic structures can degenerate to a single point in such a way that, if rescaled, it converges to a Euclidean structure. Often the same Euclidean structure is the limit of a family of degenerating spherical structures that have beenrescaled. Using the fact that the (orientation preserving) Euclidean isometry group is isomorphic to $SO(3)$ extended by translations, this can be interpreted as a transition from $SO(3,1)$ to $SO(4)$, passing through $SO(3)$ (extended) ([99], [100]).

A similar situation occurs when 3-dimensional hyperbolic structures collapse down to a hyperbolic plane. The Lie group analogy in this case suggests that there should be a transition from $SO(3,1)$ through $SO(2,1)$ (extended) to $SO(2,2)$, the AdS isometry group. Very recently ([35]), the theory of such a geometric transition has been developed. Certain hyperbolic geometry tools such as gluing ideal tetrahedra carry through the transition. This leads to a whole new class of interesting AdS structures and provides tools with which to study them.

For a closed hyperbolic 3-manifold (which has no 3-dimensional hyperbolic deformations) one can ask if it can be deformed in H^4; in other words does it have a nontrivial "quasi-Fuchsian" representation. Similar questions can be asked about deformations through projective structures or through complex hyperbolic structures. Nontrivial deformations seem to be quite rare (Cooper-Long-Thistlethwaite [31]), but manifolds that do admit nontrivial deformations are likely to be of particular interest. In such cases the character variety for representations into a larger Lie group would provide information about 3-dimensional hyperbolic geometry.

3.10. **Complex hyperbolic manifolds.** In another direction is the emerging theory of complex hyperbolic Kleinian groups. After the local rigidity theorems of Goldman [53] and Toledo [111], there were many examples of discrete groups of isometries of $\mathsf{H}^2_{\mathbb{C}}$; see for example [101]. Recent examples found by Deraux, Falbel, Paupert and Parker [39] of *non-arithmetic lattices* in $\mathsf{U}(2,1)$ were the first new examples since the original work of Deligne and Mostow in the 1980's. An important basic problem is to determine the boundary of the set of quasi-Fuchsian representations of $\pi_1(\Sigma)$ in $\mathsf{SU}(n,1)$; recent partial results have been obtained by Parker and Will.

4. Summary

There are several themes that can be extracted from the above research developments and that inform the organization of our network. Analytic problems, particularly those coming from physics, often give rise to representation spaces, providing both motivation for their study and insight into their properties. By associating geometric structures to particular types of representations, one can bring to bear many tools from topology. This is particularly true for structures on low dimensional manifolds; it is then natural to look for higher dimensional analogs of techniques constructions that prove useful in low dimensions. Utilizing dynamical systems, both those derived from flows and from discrete group actions, in the study of representation varieties can lead to deep conclusions. Conversely, such varieties often provide particularly interesting examples of dynamical systems. There are often useful relationships between different types of geometric structures and it can be valuable to look for analogous constructions in one case that have been successfully used in another. This is particularly true for combinatorial objects and invariants. These often arise from the degenerations or from asymptotic behavior.

References

[1] Agol, I., Tameness of hyperbolic 3-manifolds, math.GT/0405568.

[2] Andersen, Jørgen Ellegaard Asymptotic faithfulness of the quantum SU(n) representations of the mapping class groups. Ann. of Math. (2) 163 (2006), no. 1, 347 – 368.

[3] Andersen, Jorgen Ellegaard, Mapping Class Groups do not have Kazhdan's Property (T). GT.0706.2184

[4] Atiyah, M. F.; Bott, R. The Yang-Mills equations over Riemann surfaces. Philos. Trans. Roy. Soc. London Ser. A 308 (1983), no. 1505, 523 – 615.

[5] Atiyah, M. F.; Hitchin, N. J.; Singer, I. M. Self-duality in four-dimensional Riemannian geometry. Proc. Roy. Soc. London Ser. A 362 (1978), no. 1711, 425–461

[6] Barbot, Thierry; Charette, Virginie; Drumm, Todd; Goldman, William M.; Melnick, Karin A primer on the $(2+1)$ Einstein universe. Recent developments in pseudo-Riemannian geometry, 179–229, ESI Lect. Math. Phys., Eur. Math. Soc., Zrich, 2008

[7] Benoist, Yves Automorphismes des cônes convexes. Invent. Math. 141 (2000), no. 1, 149 – 193.

[8] Benoist, Yves Convexes divisibles. I. Algebraic groups and arithmetic, 339 – 374, Tata Inst. Fund. Res., Mumbai, 2004.

[9] Benoist, Yves Convexes divisibles. II. Duke Math. J. 120 (2003), no. 1, 97 – 120.

[10] Benoist, Yves Convexes divisibles. III. Ann. Sci. École Norm. Sup. (4) 38 (2005), no. 5, 793 – 832.

[11] Benoist, Yves Convexes divisibles. IV. Structure du bord en dimension 3. Invent. Math. 164 (2006), no. 2, 249 – 278.

[12] Bers, L., Simultaneous Uniformization, Bull. Amer. Math. Soc. 66 (1960), 94–97.

[13] Boileau, M., Leeb, B., and Porti, J., Geometrization of 3-dimensional orbifolds. Annals of Math 162 (2005), 376-421.

[14] Bonahon, Francis Geodesic laminations on surfaces. Laminations and foliations in dynamics, geometry and topology (Stony Brook, NY, 1998), 1 – 37, Contemp. Math., 269, Amer. Math. Soc., Providence, RI, 2001.

[15] Bonsante, F., Schlenker, J-M. Fixed points of compositions of earthquakes. preprint, arXiv:0812.3471.

[16] Bradlow, S., García-Prada, O., Goldman, W. and Wienhard, A., Representations of Surface Groups - Background Material for a workshop at AIM 2006, available on request.

[17] Bradlow, Steven B.; Garca-Prada, Oscar; Gothen, Peter B. Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces. Geom. Dedicata 122 (2006), 185 – 213.

[18] Bradlow, Steven B.; Garca-Prada, Oscar; Gothen, Peter B. Surface group representations and $\mathrm{U}(p,q)$-Higgs bundles. J. Differential Geom. 64 (2003), no. 1, 111 – 170.

[19] Bradlow, Steven B.; Garca-Prada, Oscar; Gothen, Peter B. Deformations of maximal representations in $\mathsf{Sp}(4,\mathbb{R})$ (submitted)

[20] Garcia-Prada, Oscar; Gothen, Peter B.; Mundet i Riera, Ignasi Higgs bundles and surface group representations in the real symplectic group, (submitted)
[21] Brock, J., Bromberg, K., Canary, R., and Minsky, Y., Local topology in deformation spaces of hyperbolic 3-manifolds, Preprint,arXiv:0911.1432.
[22] Brock, J., Canary, R., and Minsky, Y., The classification of Kleinian surface groups II: the ending lamination conjecture, Preprint, arXiv:math.GT/0412006.
[23] Brock, B., Bromberg, K. On the density of geometrically finite Kleinian groups. Acta Math. 192 (2004), no. 1, 33–93.
[24] Bromberg, K. Projective structures with degenerate holonomy and the Bers density conjecture. Ann. of Math. 166 (2007), no. 1, 77–93.
[25] Bullock, Doug; Frohman, Charles; Kania-Bartoszyska, Joanna Understanding the Kauffman bracket skein module. J. Knot Theory Ramifications 8 (1999), no. 3, 265277. (
[26] Burger, Marc; Iozzi, Alessandra; Wienhard, Anna Surface group representations with maximal Toledo invariant. Ann. of Math. (2) 172 (2010), no. 1, 517 – 566
[27] Burger, Marc; Iozzi, Alessandra; Labourie, Franois; Wienhard, Anna Maximal representations of surface groups: symplectic Anosov structures. Pure Appl. Math. Q. 1 (2005), no. 3, Special Issue: In memory of Armand Borel. Part 2, 543 – 590.
[28] Calegari, Danny; Gabai, David Shrinkwrapping and the taming of hyperbolic 3-manifolds. J. Amer. Math. Soc. 19 (2006), no. 2, 385446
[29] Charette, Virginie; Drumm, Todd A.; Goldman, William M. Affine deformations of a three-holed sphere. Geom. Topol. 14 (2010), no. 3, 1355 – 1382
[30] Choi, Suhyoung; Goldman, William M. The classification of real projective structures on compact surfaces. Bull. Amer. Math. Soc. (N.S.) 34 (1997), no. 2, 161 – 171.
[31] , Cooper, D., Long, D., Thistlethwaite, M. Flexing closed hyperbolic manifolds. Geom. Top. vol. 11 (2007), 2413-2440.
[32] Cooper, D., Hodgson, C, and Kerckhoff, S., Three-dimensional orbifolds and cone-manifolds. MSJ Memoirs, 5. Mathematical Society of Japan, Tokyo, 2000.
[33] Culler, Marc; Gordon, C. McA.; Luecke, J.; Shalen, Peter B. Dehn surgery on knots. Ann. of Math. (2) 125 (1987), no. 2, 237300.
[34] Culler, Marc; Shalen, Peter B. Varieties of group representations and splittings of 3-manifolds. Ann. of Math. (2) 117 (1983), no. 1, 109146.
[35] Danciger, J., Stanford PhD Thesis 2011.
[36] Daskalopoulos, Georgios; Weitsman, Jonathan; Wentworth, Richard; Wilkin, Graeme Cohomology of $\mathsf{SL}(2,\mathbb{C})$-character varieties of surface groups and the action of the Torelli group. math.GT.0808.0131
[37] Daskalopoulos, Georgios; Weitsman, Jonathan; Wentworth, Richard; Wilkin, Graeme Morse Theory and Hyperkahler Kirwan Surjectivity for Higgs Bundles. math/0701560
[38] Deligne, Pierre; Mostow, G. D. Monodromy of hypergeometric functions and nonlattice integral monodromy. Inst. Hautes tudes Sci. Publ. Math. No. 63 (1986), 5 – 89.
[39] Deraux, Martin; Parker, J.R.; Paupert, J., A census
[40] Donaldson, S. K. An application of gauge theory to four-dimensional topology. J. Differential Geom. 18 (1983), no. 2, 279 – 315.
[41] Drumm, Todd A. Fundamental polyhedra for Margulis space-times. Topology 31 (1992), no. 4, 677683.
[42] Ehresmann, C., Sur les espaces localement homogenes, L'ens. Math. 35 (1936), 317–333
[43] Farb, Benson and Mosher, Lee. Convex cocompact subgroups of mapping class groups. Geom. Topol. 6 (2002), 91152
[44] Fathi, Albert, Laundenbach, François, and Poénaru, Valentin, Travaux de Thurston sur les surfaces, Soc. Math. de France, 1979, Astérisque, vol. 66-67
[45] Fock, Vladimir; Goncharov, Alexander, Moduli spaces of local systems and higher Teichmüller theory. Publ. Math. Inst. Hautes Études Sci. No. 103 (2006), 1 – 211.
[46] Fomin, Sergey; Shapiro, Michael; Thurston, Dylan Cluster algebras and triangulated surfaces. I. Cluster complexes. Acta Math. 201 (2008), no. 1, 83146.
[47] Fomin, Sergey; Zelevinsky, Andrei Cluster algebras: notes for the CDM-03 conference. Current developments in mathematics, 2003, 1 – 34, Int. Press, Somerville, MA, 2003

[48] Fricke, Robert; Klein, Felix Vorlesungen über die Theorie der automorphen Funktionen. Band 1: Die gruppentheoretischen Grundlagen. Band II: Die funktionentheoretischen Ausfhrungen und die Andwendungen. (German) Bibliotheca Mathematica Teubneriana, Bnde 3, 4 Johnson Reprint Corp., New York; B. G. Teubner Verlagsgesellschaft, Stuttg art 1965

[49] Fried, David; Goldman, William M. Three-dimensional affine crystallographic groups. Adv. in Math. 47 (1983), no. 1, 1– 49.

[50] Gelander, Tsachik On deformations of F_n in compact Lie groups. Israel J. Math. 167 (2008), 15 – 26.

[51] Goldman, William M. Topological components of spaces of representations. Invent. Math. 93 (1988), no. 3, 555 – 607.

[52] Goldman, William M. Ergodic theory on moduli spaces. Ann. of Math. (2) 146 (1997), no. 3, 475 – 507.

[53] Goldman, William M. Complex hyperbolic geometry. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1999.

[54] Goldman, William M. An ergodic action of the outer automorphism group of a free group. Geom. Funct. Anal. 17 (2007), no. 3, 793805.

[55] Goldman, William M. Mapping class group dynamics on surface group representations. Problems on mapping class groups and related topics, 189 – 214, Proc. Sympos. Pure Math., 74, Amer. Math. Soc., Providence, RI, 2006

[56] Goldman, William M., Higgs bundles and geometric structures on surfaces, in The Many Facets of Geometry: a Tribute to Nigel Hitchin, (Bourgignon, Garcia-Prada & Salamon, eds.), 129 – 163, Oxford University Press 2010.

[57] Goldman, William M., Locally homogeneous geometric manifolds, Proceedings of the 2010 International Congress of Mathematicians, Hyderabad.

[58] Goldman, William M.; Labourie, François; Margulis, Gregory Proper affine actions and geodesic flows of hyperbolic surfaces. Ann. of Math. (2) 170 (2009), no. 3, 1051 – 1083,

[59] Goldman, William M.; Millson, John J. The deformation theory of representations of fundamental groups of compact Kähler manifolds. Inst. Hautes Études Sci. Publ. Math. No. 67 (1988), 43 – 96.

[60] Goldman, William M.; Xia, Eugene Z. Rank one Higgs bundles and representations of fundamental groups of Riemann surfaces. Mem. Amer. Math. Soc. 193 (2008), no. 904

[61] Peter B. Gothen, *Components of spaces of representations and stable triples*, Topology **40** (2001), no. 4, 823–850.

[62] Guichard, Olivier Composantes de Hitchin et représentations hyperconvexes de groupes de surface. J. Differential Geom. 80 (2008), no. 3, 391 – 431

[63] Guichard, Olivier; Wienhard, Anna Convex foliated projective structures and the Hitchin component for $PSL_4(\mathbf{R})$. Duke Math. J. 144 (2008), no. 3, 381 – 445.

[64] Guichard, Olivier; Wienhard, Anna Domains of discontinuity for surface groups. C. R. Math. Acad. Sci. Paris 347 (2009), no. 17-18, 1057 – 1060.

[65] Guichard, Olivier; Wienhard, Anna Topological invariants for Anosov representatons, J. Topology (to appear)

[66] Hamenstdt, U. Word hyperbolic extensions of surface groups, preprint; arXiv:math.GT/0505244. Masur

[67] Hitchin, N. J. The self-duality equations on a Riemann surface. Proc. London Math. Soc. (3) 55 (1987), no. 1, 59 – 126.

[68] Hitchin, N. J. Flat connections and geometric quantization. Comm. Math. Phys. 131 (1990), no. 2, 347 – 380.

[69] Hitchin, N. J. Lie groups and Teichmller space. Topology 31 (1992), no. 3, 449 – 473.

[70] Hodgson, C., Kerckhoff, S., Universal bounds for hyperbolic Dehn surgery.Annals of Math 162 (2005), 376-421.

[71] Hodgson, C., Kerckhoff, S. The shape of hyperbolic Dehn surgery space. Geom. Topol. 12 (2008), no. 2, 1033–1090.

[72] Jones, Vaughan F. R. A polynomial invariant for knots via von Neumann algebras. Bull. Amer. Math. Soc. (N.S.) 12 (1985), no. 1, 103111.

[73] Kent, Richard P., IV and Leininger, Christopher. Shadows of mapping class groups: capturing convex cocompactness. Geom. Funct. Anal. 18 (2008), no. 4, 12701325.

[74] Kerckhoff, Steven P. The geometry of Teichmller space. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 665 – 678, PWN, Warsaw, 1984.
[75] Labourie, François Flat projective structures on surfaces and cubic holomorphic differentials. Pure Appl. Math. Q. 3 (2007), no. 4, part 1, 1057 – 1099.
[76] Labourie, François Anosov flows, surface groups and curves in projective space. Invent. Math. 165 (2006), no. 1, 51 – 114.
[77] Labourie, Franois Cross ratios, Anosov representations and the energy functional on Teichmller space. Ann. Sci. c. Norm. Supr. (4) 41 (2008), no. 3, 437 – 469.
[78] Loftin, John C. Affine spheres and convex $\mathbb{RP}^n$-manifolds. Amer. J. Math. 123 (2001), no. 2, 255 – 274.
[79] Loftin, John C. The compactification of the moduli space of convex $\mathbb{RP}^2$ surfaces. I. J. Differential Geom. 68 (2004), no. 2, 223 – 276.
[80] Lusztig, G. Total positivity in reductive groups. Lie theory and geometry, 531 – 568, Progr. Math., 123, Birkhäuser Boston, Boston, MA, 1994
[81] Margulis, G. A. Discrete groups of motions of manifolds of nonpositive curvature. (Russian) Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), Vol. 2, pp. 21 – 34. Canad. Math. Congress, Montreal, Que., 1975.
[82] Margulis, G. A. Complete affine locally flat manifolds with a free fundamental group. Automorphic functions and number theory, II. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 134 (1984), 190 – 205.
[83] Masur, Howard. Measured foliations and handlebodies. Ergodic Theory Dynam. Systems 6 (1986), no. 1, 99116.
[84] McCarthy, John and Papadopoulos, Athanase. Dynamics on Thurston's sphere of projective measured foliations. Comment. Math. Helv. 64 (1989), no. 1, 133166.
[85] Mess, G. Lorentz spacetimes of constant curvature. Geom. Dedicata 126 (2007), 3–45.
[86] Milnor, John On fundamental groups of complete affinely flat manifolds. Advances in Math. 25 (1977), no. 2, 178 – 187.
[87] Milnor, J. Hilbert's problem 18: on crystallographic groups, fundamental domains, and on sphere packing. Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Northern Illinois Univ., De Kalb, Ill., 1974), pp. 491 – 506. Proc. Sympos. Pure Math., Vol. XXVIII, Amer. Math. Soc., Providence, R. I., 1976.
[88] Minsky, Yair The classification of Kleinian surface groups. I. Models and bounds. Ann. of Math. (2) 171 (2010), no. 1, 1 – 107
[89] Minsky, Yair Note on dynamics of $\mathsf{Out}(\mathbb{F}_n)$ on $\mathrm{PSL}(2,\mathbb{C})$-characters, arXiv:0906.3491.
[90] Morgan, John W.; Shalen, Peter B. Valuations, trees, and degenerations of hyperbolic structures. I. Ann. of Math. (2) 120 (1984), no. 3, 401476.
[91] Morgan, John W.; Shalen, Peter B. Degenerations of hyperbolic structures. III. Actions of 3-manifold groups on trees and Thurston's compactness theorem. Ann. of Math. (2) 127 (1988), no. 3, 457519.
[92] Mostow, G. D. Strong rigidity of locally symmetric spaces. Annals of Mathematics Studies, No. 78. Princeton University Press, Princeton, N.J.
[93] Narasimhan, M. S.; Seshadri, C. S. Stable and unitary vector bundles on a compact Riemann surface. Ann. of Math. (2) 82 1965 540 – 567.
[94] Penner, R. C. The decorated Teichmller space of punctured surfaces. Comm. Math. Phys. 113 (1987), no. 2, 299339.
[95] Pickrell, Doug; Xia, Eugene Z. Ergodicity of mapping class group actions on representation varieties. I. Closed surfaces. Comment. Math. Helv. 77 (2002), no. 2, 339 – 362
[96] Pickrell, Doug; Xia, Eugene Z. Ergodicity of mapping class group actions on representation varieties. II. Surfaces with boundary. Transform. Groups 8 (2003), no. 4, 397 – 402.
[97] Poincaré, H. Sur les fonctions abéliennes. Acta Math. 26 (1902), no. 1, 43 – 98
[98] Poincaré, H. Mémoire. Les groupes kleinéens. Acta Math. 3 (1966), no. 1, 49 – 92,
[99] Porti, J. Regenerating hyperbolic and spherical cone structures from Euclidean ones. Topology 37 (1998), 365-392.
[100] Porti, J. and Weiss, H. Deforming Euclidean cone 3-manifolds. Geom. Topol. 11 (2007), 1507–1538.

[101] Schwartz, Richard Evan Spherical CR geometry and Dehn surgery. Annals of Mathematics Studies, 165. Princeton University Press, Princeton, NJ, 2007.

[102] Simpson, Carlos T. Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization. J. Amer. Math. Soc. 1 (1988), no. 4, 867 – 918

[103] Simpson, Carlos T. Nonabelian Hodge theory. Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 747 – 756, Math. Soc. Japan, Tokyo, 1991.

[104] Sikora, Adam S. SL_n-character varieties as spaces of graphs. Trans. Amer. Math. Soc. 353 (2001), no. 7, 27732804

[105] Taubes, Clifford Henry Self-dual Yang-Mills connections on non-self-dual 4-manifolds. J. Differential Geom. 17 (1982), no. 1, 139 – 170.

[106] Thurston, William P. On the geometry and dynamics of diffeomorphisms of surfaces. Bull. Amer. Math. Soc. (N.S.) 19 (1988), no. 2, 417 – 431

[107] Thurston, William P. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. Bull. Amer. Math. Soc. (N.S.) 6 (1982), no. 3, 357 – 381.

[108] Thurston, William P. Hyperbolic structures on 3-manifolds. I. Deformation of acylindrical manifolds. Ann. of Math. (2) 124 (1986), no. 2, 203246

[109] Thurston, William P. Three-dimensional geometry and topology. Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997

[110] Thurston, W. The Geometry and Topology of three-manifolds. Princeton Univ. Math. Dept. Notes (1979).

[111] Toledo, Domingo Representations of surface groups in complex hyperbolic space. J. Differential Geom. 29 (1989), no. 1, 125 – 133.

[112] Uhlenbeck, Karen K. Connections with L^p bounds on curvature. Comm. Math. Phys. 83 (1982), no. 1, 31–42.

[113] Vogt, H., Sur les invariants fondamentaux des équations différentielles linéaires du second ordre. Ann. Sci. École Norm. Sup. (3) 6 (1889), 3–71.

[114] Weiss, H. Local rigidity of 3-dimensional cone-manifolds. JDG 71 (2005), no. 3, 437–506.

[115] Wienhard, Anna, The action of the mapping class group on maximal representations. Geom. Dedicata 120 (2006), 179 – 191.