The minimal surface is unique

(sometimes)

Let S a connected closed surface of genus $g \geq 2$.

Theorem Let G be real Lie group $\boxed{\mathrm{rk}\, G = 2}$, $\rho : \pi_1 S \to G$. If

1. G split & ρ Hitchin, or (Labourie)
2. G hermitian & ρ maximal (C-Tholozan-Toulisse).

Then $\underline{\exists !}$ $X_\rho \in \mathrm{Teich}(S)$ so that the harmonic metric

$h_\rho : \tilde{X}_\rho \to G/K$ is conformal. $\rightsquigarrow$ h_ρ a minimal surface.

In other words $\exists!$ critical point $\mathcal{E}_\rho : \mathrm{Teich}(S) \to \mathbb{R}$.

Corollary $\pi : \mathrm{Hit}(G) \to \mathrm{Teich}(S)$ hol. vector bundle.

$\pi : \mathrm{Max}(G) \to \mathrm{Teich}(S)$ hol fibration.

Mod(S)-equivariant parameterizations

$\pi^{-1}(X) \cong \mathcal{M}^{Higgs}_{K^j}(G_c)$ where G_c is a rank 1 real Lie group.

We will see how the proof uses $\boxed{\mathrm{rk}(G) = 2}$ in an essential way.

1. $G = \underbrace{(P)SL_2\mathbb{R}}_{\text{Schoen}}{}^{1} \times (P)SL_2\mathbb{R}$, $\underbrace{SL_3\mathbb{R}}_{\text{Loftin \& Labourie}}{}^{2}$, $(P)Sp(4,\mathbb{R})^{3}$, $G_2^{split\,4}$

$G_c = \mathbb{R}^+$ $\mathcal{M}_{K^j}(X, \mathbb{R}^+) = H^0(K^j) = \boxed{Q_j(j)}$

$j = 2, \boxed{3}, 4, 6.$

Note. for $G = PSL_2\mathbb{R} \times PSL_2\mathbb{R}$. $\mathrm{Hit}(G) = \mathrm{Teich}(S) \times \mathrm{Teich}(S)$.

$\rho \in \mathrm{Hit}(G) \rightsquigarrow X_1, X_2 \in \mathrm{Teich}(S)$ $\exists!$ $X_\rho \to X_1 \times X_2$.

2. $\mathfrak{g} = \mathfrak{so}(2,n)$, $\mathfrak{su}(1,n) \times \mathfrak{su}(1,m)$, $\mathfrak{su}(2,n)$, $\mathfrak{so}^*(8)$, $\mathfrak{so}^*(10)$, $\underbrace{\mathfrak{sp}(4,\mathbb{R})}_{\text{also C, Alessandrini-C}}$

* For the theorem it suffices to consider $G = SO_0(2,n)$. *

$G_c = SO(1,n-1)$ $j = 2$. $\pi^{-1}(X) = \mathcal{M}_{K^2}(SO(1,n-1))$.

What is a maximal rep?

G-Hermitian $\iff$ G/K - Kähler $\iff$ $U(1) = Z(K) < K$.

$\rho : \pi_1 S \to G$ attach an $\mathbb{Z}$-invariant $\tau(\rho)$.

$$|\tau(\rho)| \leq rk(G)(2g-2).$$

$$Max(G) = \{ \rho : \pi_1 S \to G \mid \tau(\rho) = rk(G)(2g-2) \} / G.$$

Corollary of Markovich: G hermitian, $rk(G) \geq 3$,

$\rho : \pi_1 S \to G$ max. $\Rightarrow$ statement of Thm. fails.

Remark: For $G = SO(p,q)$ $1 < p < q$ or G quaternionic real form of F_4, E_6, E_7, E_8
The statement of Thm may still hold for "positive reps".

Key Ingredients of Proof of Theorem.

- Existence (Labourie, Burger-Iozzi-Labourie-Wienhard)
  All $\rho \in Hit(G)$ and all $\rho \in Max(G)$ are Anosov

  $\overset{\text{Labourie}}{\Longrightarrow}$ $\underbrace{\mathcal{E}_\rho : Teich(S) \longrightarrow \mathbb{R}}_{\rho \text{ harmonic map } h_\rho : \hat{X} \to G/K.}$ is proper, $\mathcal{E}_\rho(X) = \int_X \|dh_\rho\|$

- Higgs bundles: The Higgs bundle ass. to ρ on a Riemann surface X which is a critical point of $\mathcal{E}_\rho$ is a fixed point of a $\mathbb{Z}/n\mathbb{Z}$-action.

  Work Baraglia $\leadsto$ a holomorphic splitting is unitary.

  $\leadsto$ • An extra special surface $f_\rho : \hat{X}_\rho \to G/H$ ← another homogeneous space.
  which is unique $\leadsto$ uniqueness of h_ρ.

Remark: These tools can be used to prove some results in higher rank but not for all reps in certain components.
only for the cyclic locus.

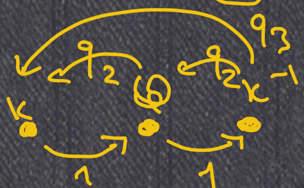
1. Fixed by $\mathbb{Z}/\boxed{j}\mathbb{Z}$ for G-simple. $\pi^{-1}(X) = H^0(K^{j})$

For $\rho : \pi_1 S \to SL_3\mathbb{R}$ Hitchin & $X \in \mathrm{Teich}(S)$.

The Higgs bundle ass. to ρ on X has the form.

$$\mathcal{E} = K \oplus \mathcal{O} \oplus K^{-1} \qquad \Phi : \mathcal{E} \to \mathcal{E} \otimes K$$

$$\begin{pmatrix} 0 & q_2 & \boxed{q_3} \\ 1 & 0 & q_2 \\ 0 & 1 & 0 \end{pmatrix} \qquad q_i \in H^0(K^i)$$



$G/T \quad T < K$ maximal torus

$$\begin{tikzcd} & SL_3\mathbb{R}/SO(2) \arrow[d] \\ X \arrow[ur, "f_\rho"] \arrow[r] & SL_3\mathbb{R}/SO(3) \end{tikzcd}$$

$h_\rho : X \to SL_3\mathbb{R}/SO(3)$ harmonic. h_ρ is minimal. $\Longleftrightarrow$ $q_2 = 0$

$\Longleftrightarrow$ fixed point of $\mathbb{Z}/3\mathbb{Z}$. action.

f_ρ is a cyclic surface. (very special) $\rightsquigarrow$ f_ρ is rigid. $\to$ uniqueness of h_ρ.

$SL_3\mathbb{R}/T = SL_3\mathbb{R}/SO(2) \qquad (\mathbb{R}^3, g) = \boxed{\mathbb{R}} \oplus \mathbb{R}^2$

proj. onto $\mathbb{R}$ gives a map to $\mathbb{RP}^2$